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VOLUME XII

1

On Spherical Waves of Finite Amplitude produced by the sudden explosion of a detonating gas contained within a spherical envelope.

By

Sudhansukumar Banerji.

[*Read March 13th, 1921.*]

Mach and Sommer announced long ago in 1877 from experimental observations that the velocity of an explosion sound wave depends upon its intensity, the method of its production, and the distance from the source. Wolff [1] sought to determine the nature of the disturbance resulting from an explosion and concluded that the type is essentially that of a sound wave the velocity increasing with the violence of the explosion. Vielle[2] had previously found that he could obtain a wave velocity in air of 1,100 metres per second by sharpening the explosion

[1] *Wied. Ann.*, 69, 329, 1899.

[2] *Compt. Rendus*, 127, 41, 1898.

of fulr ate by tightly packing it. Owing to its importance in war, sound anging for instance, the velocity of sound produced by firing large guns has been studied extensively by military as well as other investigators. But the data so obtained have not been available to the present writer.

In a previous paper [1] published in this Bulletin, I found by following Riemann's method that the velocity of propagation of spherical waves of finite amplitude is

$$\pm\left[\frac{1}{\rho}\cdot\frac{dp}{d\log\rho r^2}\right]^{\frac{1}{2}}+q,$$

for the forward and the backward motion, where q denotes the particle velocity, r the radius vector from the source of disturbance, p the pressure and ρ the density. This expression can also be written in the form

$$\pm\left[\frac{dp}{d\rho}\,\frac{1}{1+2\frac{dr}{d\rho}\cdot\frac{\rho}{r}}\right]^{\frac{1}{2}}+q.$$

Obviously this expression depends on the distance and the density or the pressure, if the pressure and the density be supposed to be connected either according to Boyle's law or the adiabatic law.

As the pressure at any point will depend on the violence of the explosion, if by any means we can determine it, then the velocity of propagation also becomes determinable. We know from hydrodynamics that for impulsive generation of motion if P be the impulsive pressure at any point, then

$$u=-\frac{1}{\rho}\frac{\partial P}{\partial x},\ v=-\frac{1}{\rho}\frac{\partial P}{\partial y},\ w=-\frac{1}{\rho}\frac{\partial P}{\partial z},$$

assuming that the medium was initially at rest. The equation of continuity is

$$\frac{\partial\rho}{\partial t}+\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}+\frac{\partial(\rho w)}{\partial z}=0.$$

[1] *Bull. Cal. Math. Soc.*, 11, 83, 1920.

This gives

$$\frac{\partial \rho}{\partial t}=\frac{\partial^2 P}{\partial x^2}+\frac{\partial^2 P}{\partial y^2}+\frac{\partial^2 P}{\partial z^2}.$$

If the gas be supposed to obey Boyle's law, then

$$P=c^2\rho,$$

therefore $$\frac{\partial P}{\partial t}=c^2\left(\frac{\partial^2 P}{\partial x^2}+\frac{\partial^2 P}{\partial y^2}+\frac{\partial^2 P}{\partial z^2}\right).$$

The explosion may be regarded as communicating a sudden pressure of magnitude Π (say) over the sphere $r=a$ and therefore as an instantaneous source of disturbance of magnitude Π. The above equation can be written in the form

$$\frac{\partial (rP)}{\partial t}=c^2\ \frac{\partial^2 (rP)}{\partial r^2}.$$

From the theory of conduction[1] of heat, we know that the solution of this equation can be taken in the form

$$P=\frac{\Pi}{(2c\sqrt{\pi t})^3}\ e^{-\frac{r^2}{4c^2 t}}.$$

Since this expression vanishes at $t=0$ except at the origin, and

$$\iiint P\,dx\,dy\,dz$$

is equal to Π for $t>0$, the above expression for P represents the disturbance communicated by such a source of magnitude Π. This expression for P when substituted in the expression for the velocity of propagation will give the velocity of propagation of the explosion waves at any point.

[1] Carslaw's *Fourier series and Integrals*, p. 347

Arthur Foley in a recent paper published in the *Physical Review* for 1920 has described an experimental method for determining the velocity of propagation of spark waves which have all the characteristics of waves of finite amplitude. This paper however does not contain any data which can be made use of to verify the above results. The author intends when suitable data are available to subject the above results to experimental verification.

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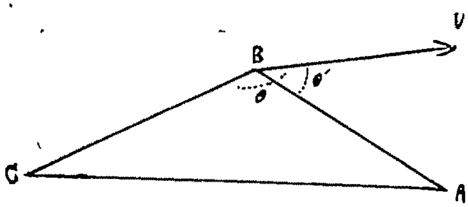


Fig. 1.

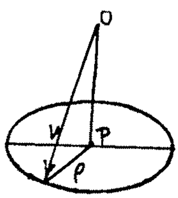
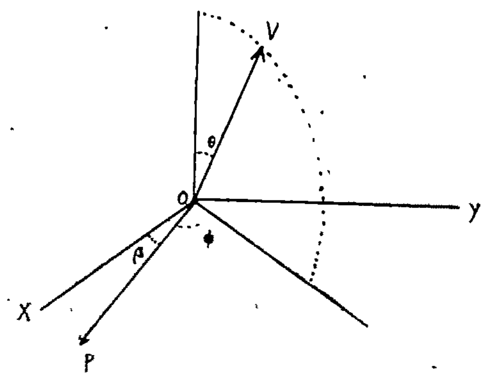


Fig 3.



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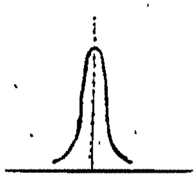

Fig 4.

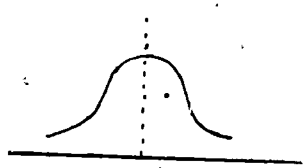

Fig 5.

Illustrating Mr. Panchanan Das's paper on the Scattering of light by molecules in rapid motion.

2

ON THE SCATTERING OF LIGHT BY MOLECULES IN RAPID MOTION.[1]

BY

PANCHANAN DAS.

[*Read March 13th, 1921.*]

In a note published in Nature (May, 1919) Raman and Larmor showed that when light is scattered by particles in rapid motion, the scattered light undergoes a change in frequency due to two causes (1) the motion of the particles relative to the source (2) the motion of the particles relative to the observer. Strutt[2] recently described an experiment which definitely established the fact that it is actually the molecules of air that take part in scattering light and producing the blue colour of the sky. The object of this paper is to calculate the change in frequency and the intensity of light scattered by such moving molecules or particles.

We take a fixed system of axes (OX, OY, OZ) with the source at the origin and the X-axis parallel to the velocity v of the molecule. We also take a moving system of axes (O'X', O'Y', O'Z') which is at rest relative to the molecule. If the two systems coincide at the instant $t=0$, the relativistic equations of transformation of coordinates are

$$t=\beta\left(t'+\frac{vx'}{c^2}\right),\ x=\beta(x'+vt'),\ y=y',\ z=z'$$

where x, y, z and x', y', z' are the coordinates of the molecule referred to the fixed and moving axes respectively and $\beta=\dfrac{1}{\sqrt{1-\dfrac{v^2}{c^2}}}$ and $c=$ velocity of light. If $r^2=x^2+y^2+z^2$ and $r'^2=x'^2+y'^2+z'^2$, then

$$r=r'+\frac{1}{r'}\left(\beta vx't'-\frac{v^2x'}{2c^2}\right), \text{ approximately.}$$

[1] Communicated by Prof. C. V. Raman, M.A.

[2] Proc. Royal Society, Vol. 94, 1918.

if we consider t' so small that t'^2 can be neglected. The disturbance from the source may be represented by.

$$S=\frac{A}{r}\sin 2\pi\left(\nu t-\frac{r}{\lambda}\right)$$

Making the above-mentioned substitution, we get

$$S=\frac{A}{r'}\sin\ 2\pi\left(\nu't'-\frac{r'}{\lambda}\right)\text{approximately.}$$

where $$\nu'=\left(\nu-\frac{v}{\lambda}\cdot\frac{x'}{r'}\right)\beta.$$

Thus ν' is the frequency of radiation apparent to the molecule.

At the instant $t=0$. $\frac{x'}{r'}=\beta\frac{x}{r}=\cos\theta$, say

Hence $$\nu'=\beta\left(\nu-\frac{v\cos\theta}{\lambda}\right)$$

or $$\frac{\nu'}{\nu}=1+\frac{v\cos\theta}{\lambda}\qquad\ldots\quad(1)$$

Now the stationary observer would appear to be moving with velocity $-v$ referred to the axes which are fixed relative to the molecule. Therefore, if ν'' be the frequency apparent to the observer,

$$\frac{\nu'}{\nu''}=1-\frac{v\cos\theta'}{c}\qquad\ldots\quad(2)$$

where θ' is the angle made by the radius vector from the molecule to the observer with the X-axis.

Multiplying (1) and (2) we get

$$\frac{\nu}{\nu''}=1-\frac{v}{c}\,(\cos\theta'-\cos\theta)$$

Let A be the observer, B the molecule, and C the source.

Using slightly different notations (see fig. 1), we get

$$\frac{n}{n'}=1-\frac{V}{c}\,(\cos\theta+\cos\theta').\qquad\ldots\quad(3)$$

where V is the velocity of molecule, n, n' are the original and altered frequencies respectively.

To determine the intensity of radiation of a particular frequency we first find the number of molecules, which emit it. From the kinetic theory of gases the number N of molecules, of which the velocity components lie between u, v, w, and $u+du$, $v+dv$, $w+dw$ is proportional to

$$e^{-hm(u^2+v^2+w^2)} du dv dw. \qquad \ldots \quad (4)$$

where h is a function of the temperature of the gas and m the mass of a molecule of the gas.

The number of molecules, for which n' is constant is found by integrating (4) over the region (3).

We take the instantaneous position of the molecule as the origin of coordinates, and the direction of the source as X-axis. The plane XY is determined by the primary light and the direction of the observer P.

Resolve the velocity V along OX, OY, OZ. The components are

$$\bar{u}, \bar{v}, \bar{w}, = V \sin\theta \cos\phi, V \sin\theta \sin\phi, V \cos\theta.$$

If the direction of the observer makes an angle β with OX, then (3) reduces to

$$\frac{n}{n'} = 1 - \frac{1}{c} \left[(1 + \cos\beta)\, \bar{u} + \bar{v} \sin\beta \right] \qquad \ldots \quad (5)$$

This is a plane parallel to the Z-axis. So the integral of (4) is a surface integral over a plane.

The equation (5) can be written in the form

$$\bar{u} \cos\frac{\beta}{2} + \bar{v} \sin\frac{\beta}{2} = \frac{c\left(1 - \frac{n}{n'}\right)}{2\cos\frac{\beta}{2}} = p. \qquad \ldots \quad (6)$$

Let P be the point (Fig 3.) where the perpendicular from O meets this plane. Let ρ be the radius of a circle with P as centre in this plane. Then

$$\rho^2 = V^2 - p^2.$$

If we take the annular space between the circles of radii ρ and $\rho+d\rho$ as the element of surface, then

$$\text{N varies as} \int_0^\infty e^{-hm V^2} 2\pi\rho d\rho,$$

or

$$N = C\, e^{-\kappa(n'-n)^2} \qquad \ldots \quad (7)$$

where

$$k = \frac{hmc^2}{n'^2 \cos^2 \frac{\beta}{2}}. \qquad \ldots \quad (8)$$

Now Rayleigh's[2] formula for the intensity of scattered light in a direction making an angle β with the primary beam of light is

$$A^2 \left(\frac{D'-D}{D}\right)^2 (1+\cos^2\beta) \frac{N^2 . \pi^2 T^2}{\lambda^4 r^2}.$$

where A is the amplitude of vibration, D, D′ the original and altered densities of ether, λ the wave length, r the distance of the molecule from the observer, and T the volume occupied by a molecule.

Substituting the value of N from (7) we get

$$\text{Intensity } I = C.\, A^2 \left(\frac{D'-D}{D}\right)^2 (1+\cos^2\beta)\, \frac{n'^4}{c^4} \cdot \frac{\pi^2 T^2}{r^2} \cdot e^{-k(n'-n)^2} \qquad \ldots \quad (9)$$

Hitherto we have assumed that the primary light is monochromatic. If it is due to a real spectrum line, the light is not strictly monochromatic, but the intensity of different parts of the spectrum is given by Maxwell's law :—

$$A^2 = C_0\, e^{-\mu(n-n_0)^2} \qquad \ldots \quad (10)$$

where μ is a very small quantity of the same order as hm, and n_0 is the frequency of the central part of the line.

Hence the intensity must be found by substituting (10) in (9) and integrating (9) with respect to n from 0 to ∞ and taking twice the value of the integral. Thus,

$$I = 2\, C_0 C \left(\frac{D'-D}{D}\right)^2 \frac{(1+\cos^2\beta)\pi^2 T^2 n'^4}{c^4 r^2}.$$

$$\int_0^\infty e^{-\mu(n-n_0)^2}\, e^{-k(n'-n)^2}\, dn.$$

Now
$$\mu(n-n_0)^2+k(n'-n)^2$$
$$=(\mu+k)\left(n-\frac{\mu n_0+kn'}{\mu+k}\right)^2+\frac{k\mu}{k+\mu}(n_0-n')^2,$$

therefore
$$\int_0^\infty e^{-\mu(n-n_0)^2-k(n'-n)^2}\,dn$$

$$=e^{-\frac{k\mu}{k+\mu}(n_0-n')^2}\int_0^\infty e^{(z-z_0)^2(\mu+k)}\,dz$$

$$=e^{-\frac{k\mu}{k+\mu}(n_0-n')^2}\left[\int_0^{z_0} e^{(\mu+k)\,(z-z_0)^2}\,dz+\int_{z_0}^\infty e^{(\mu+k)\,(z-z_0)^2}\,dz\right]$$

$$=e^{-\frac{k\mu}{k+\mu}(n_0-n')^2}\left[\int_{-z_0}^{0} e^{(\mu+k)^2z^2}\,dz+\int_0^\infty e^{(\mu+k)z^2}\,dz\right]$$

$$=e^{-\frac{k\mu}{k+\mu}(n_0-n')^2}\left[\int_0^\infty e^{(\mu+k)z^2}\,dz+\int_0^\infty e^{(\mu+k)z^2}\,dz\right]$$

approximately, if we take z_0 to be very large.

Thus $I=\ K\ e^{-\frac{\mu k}{\mu+k}(n'-n_0)^2}$.

Now μ is of the order hm. while $k=\dfrac{hmc^2}{n'^2\cos\frac{\beta}{2}}$. Thus the intensity is a function of the direction of observation, and as k is much smaller than μ, the energy distribution in the original spectrum line and that

in the scattered light must be different. Figs. 4 and 5 represent the energy-curves of the primary and scattered light respectively. Apparently the width of the spectrum-line becomes larger by scattering.

My thanks are due to Prof. C. V. Raman for the helpful interest he has taken in this paper.

3

Radial strain in a grvitating sphere

By

Jyotirmaya Ghosh

[*Read September 4th, 1920.*]

1. When a homogeneous elastic sphere is strained by the mutual gravitation of its parts, there exists within the sphere a spherical surface over which the displacement is zero and on one side of this sphere of no displacement, the displacement is contraction and on the other it is extension.[1]

It will be shown here that the same remark applies also in the following cases :—

(1) A homogeneous sphere with a concentric spherical hollow or nucleus with a density different from the rest of the sphere.

(2) A heterogeneous sphere, the density at any point varying as some positive integral power of its distance from the centre. The sphere may be either complete or hollow, the latter case giving an exception when the law of density is that of direct distance.

2. The displacement equation of equilibrium is

$$(\lambda+2\mu)\ \frac{\partial}{\partial r}\left(\frac{\partial U}{\partial r}+\frac{2U}{r}\right)+\rho R=0, \qquad \dots \quad (1)$$

where U is the radial displacement and R is the force of attraction at any point.

The solution of the equation is clearly

$$U=Ar+\frac{B}{r^2}-\frac{1}{\lambda+2\mu}\cdot\frac{1}{r^2}\int[r^2\textstyle\int\rho R dr]dr. \qquad \dots \quad (2)$$

3. In the case of a homogeneous thick shell bounded by $r=r_1$ and $r=r_0$ the attraction at any point distant r from the centre ($r_0>r>r_1$) is

$$R=-\kappa\rho\left(r-\frac{r_1^3}{r^2}\right), \qquad \dots \quad (3)$$

[1] See Love's Elasticity, (second edition), p. 140.

where the constant of gravitation is put$=1$, ρ is the density of the material and $\kappa=\frac{4\pi}{3}$.

Substituting (3) in (2) we get on integration

$$U=-\frac{\kappa\rho^2}{\lambda+2\mu}r_1^3+Ar+\frac{B}{r^2}-\frac{\kappa\rho^2}{10(\lambda+2\mu)}\cdot r^3,$$

$$=Ar+\frac{B}{r^2}+a+br^3, \qquad \dots \quad (4)$$

where $a=-\frac{\kappa\rho^2}{\lambda+2\mu}$, $b=\frac{-\kappa\rho^2}{10(\lambda+2\mu)}$.

The equation (4) in U and r is of the fifth degree, so that there is certainly one real root of the equation $U=0$ different from zero. Hence there must be one sphere of no displacement.[1]

Now, suppose that the sphere of radius $r=r_0$ has a nucleus in the form of a concentric sphere of radius $r=r_1$ and density ρ'.

From any point in the inner sphere the attraction is the same as if the sphere were homogeneous and of density ρ' throughout. At any point between the nucleus and the outer boundary, we have

$$R=-\kappa\left[\rho r+\frac{r_1^3(\rho'-\rho)}{r^2}\right]. \qquad \dots \quad (5)$$

which is of the same form as (3). Hence the solution will be of the same from as (4) and we shall have b' instead of (b) where b'

$$=\frac{\kappa\rho(\rho'-\rho)}{10(\lambda+2\mu)}.$$

4. When the sphere is *heterogeneous*, the density at any point being cr^n where c is constant, r the distance of the point from the centre and n a positive integer.

(1) *Complete sphere.*

The attraction at any point distant r from the centre ($r<r_0$) is given by

$$R=-\frac{4\pi c}{n+3}r^{n+1}. \qquad \dots \quad (6)$$

[1] This statement is true only if the root lies between r_0 and r_1, that is to say, the equation undergoes a change in sign between r_0 and r_1. Whether this condition can be fulfilled or not will obviously depend on A and B, that is to say, on the nature of the tractions at the free surfaces.—S. K. B.

Therefore $$\rho R = -\frac{4\pi c^2}{n+3} r^{2n+1}.$$

Substituting this in (2) and integrating

$$U = Ar + \frac{B}{r^2} + \frac{2\pi c^2}{(n+1)(n+3)(2n+5)(\lambda+2\mu)} \cdot r^{2n+3}$$

$$= Ar + \frac{B}{r^2} + c' \ r^{2n+3}, \qquad \dots \quad (7)$$

where $$c' = \frac{2\pi c^2}{(n+1)(n+3)(2n+5)(\lambda+2\mu)}.$$

As the sphere is complete we must take $B=0$ in the solution (7).

Putting $U=0$, the equation (7) becomes a binomial equation of even degree which must have a real positive root provided $\frac{A}{c'}$ is negative. And this is actually the case both when the sphere is under constant external pressure and when it is free. Hence there exists a sphere of no displacement.

In the case of a free sphere, when the law of density is that of direct distance, the radius of this sphere is found to be

$$r = r_0 \sqrt[4]{\frac{2(3\lambda+4\mu)}{3\lambda+2\mu}}.$$

When the sphere of radius $r=r_0$ has a concentric hollow of radius $r=r_1$, we have

$$R = -\frac{4\pi c}{n+3}\left(r^{n+1} - \frac{r_1^{n+3}}{r^2} \right), \qquad \dots \quad (8)$$

therefore $$\rho R = -\frac{4\pi c^2}{n+3}\left(r^{2n+1} - r_1^{n+1} \cdot r^{n-2} \right).$$

Substituting this in (2) we get

$$U = Ar + \frac{B}{r^2} + c' r^{2n+3} + c'' r^n, \qquad \dots \quad (9)$$

where $$c'' = \frac{4\pi c^2}{(n-1)(n+2)(n+3)(\lambda+2\mu)}.$$

Since equation (9) is of an odd degree, it must have a real root, and therefore there is a sphere of no displacement.

The solution (9) evidently fails when $n=1$.

5. It should be noticed that the solutions (7) fail to give the displacement when n has either of the values -1, -3, or $-\frac{5}{2}$. The density in all these cases also become infinite at the origin. We therefore proceed independently by assuming the centre to be enclosed by a concentric spherical hollow of radius a. The solutions are found to be as follows:—

(a) When $n=-1$, $\rho=cr^{-1}$ we have $R=-2\pi(c-a)$

and
$$U=Ar+\frac{B}{r^2}+\frac{1}{\lambda+2\mu}\cdot\frac{2\pi c^2}{a}\left[(3\log r-1)r+\frac{9a^2}{2r}\right].$$

(b) When $n=-3$, $\rho=cr^{-3}$, $R=-\frac{4\pi c}{rc^2}\log\frac{r}{a}$

and
$$U=Ar+\frac{B}{r^2}+\frac{1}{\lambda+2\mu}\,\frac{\pi c^2}{r^3}\left(\log\frac{r}{a}+\frac{5}{4}\right).$$

(c) When $n=-\frac{5}{2}$, $\rho=cr^{-5/2}$, $R=-\frac{8\pi c}{r^2}(r^{1/2}-a^{1/2})$

and
$$U=Ar+\frac{B}{r^2}+\frac{1}{\lambda+2\mu}\cdot\frac{8\pi c^2}{r^2}\left[-\frac{1}{3}\log r+\frac{4a^{\frac{1}{2}}}{7r^{\frac{1}{2}}}\right].$$

Similarly, the solution (9) when $n=1$, takes the form

$$U=Ar+\frac{B}{r^2}+\frac{\pi c^2}{28(\lambda+2\mu)}\,r^5+\frac{\pi c^2}{3(\lambda+2\mu)}\left(\log r-\frac{1}{3}\right)r^3.$$

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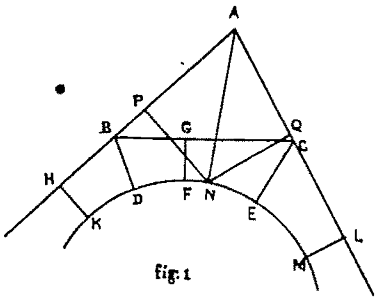

fig. 1

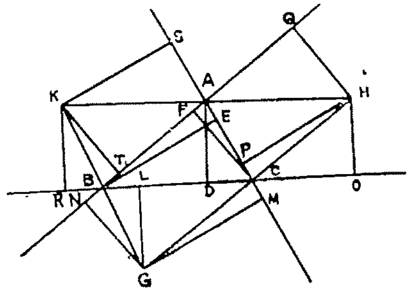

fig. 2.

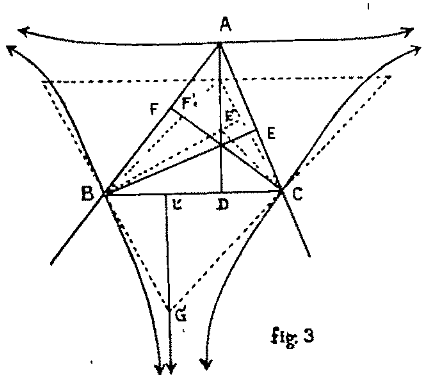

fig. 3

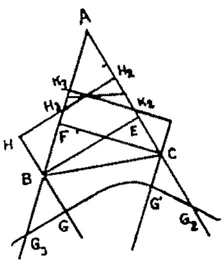

fig. 4

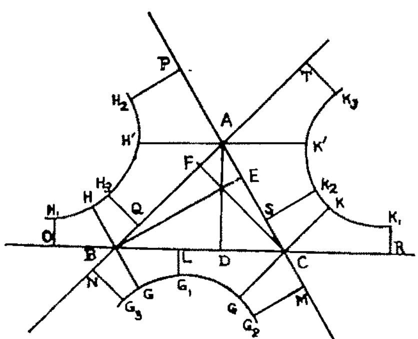

fig. 5.

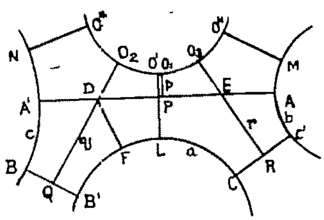

fig. 6.

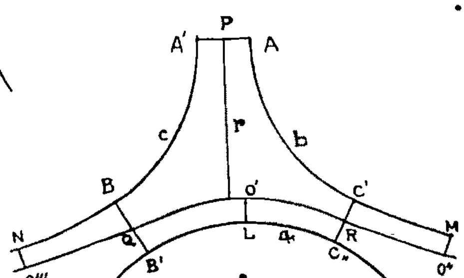

fig. 7.

Illustrating Mukhopadhyaya and Bhar's paper on generalisation of certain theorems in the hyperbolic geometry of the triangle.

4

Generalisation of certain theorems in the Hyperbolic Geometry of the Triangle

By

S. Mukhopadhyaya and G. Bhar.[1]

[*Read March 9th, 1919.*]

Introduction.

The geometry of the triangle on the hyperbolic plane has many remarkable features which are absent in the geometry of the plane triangle and which are brought out the more prominently by a purely geometrical treatment. We will consider two well-known theorems in the geometry of the hyperbolic triangle with a view to elegant geometrical demonstrations and extensions to the case where one or more of the vertices are ideal or improper points. In the course of the investigations we will come to some very remarkable new theorems. It is hoped that the present paper will prove interesting to lovers of pure Non-Euclidean Geometry.

We have in Euclidean geometry the two well-known theorems : (*i*) The three internal bisectors of the angles of a triangle or two external and one internal bisector meet at a point. (*ii*) The three perpendiculars on the sides of a triangle from the opposite vertices meet at a point.

We will discuss their analogues on the hyperbolic plane with actual, ideal or improper vertices.

A system of lines on a hyperbolic plane are said to meet at an ideal point when they are all perpendicular to the same straight line. This straight line is uniquely representative of the ideal point. The system of lines are said to meet at an improper point when they are parallel to one another in the same sense.

Theorem I:—*The three internal bisectors of the angles of a hyperbolic triangle ABC meet at an actual point.*

The internal bisector of an angle A must meet the opposite side at some point D. The internal bisector of B will meet AD at some point

[1] This paper was read before the Calcutta Mathematical Society in an abstract form. I owe to my pupil Mr. G Bhar, M.Sc., the present expanded form of my paper embracing all the different cases and the carefully drawn diagrams.—S M.

O. The perpendiculars from O on AC and BC are each equal to the perpendicular from O on AB Therefore the internal bisector of the angle C passes through O.

Theorem II:—*The external bisectors of any two angles B and C of a hyperbolic triangle ABC meet the internal bisector of the third angle A at an actual, ideal or improper point.*

If any two of the three bisectors pass through an actual point the third can be shown to pass through the same actual point as in Theorem I.

If no two of the three bisectors meet at an actual point, then the two external bisectors of the angles B and C either meet at an ideal point or are parallel.

Suppose the two external bisectors BD and CE meet at an ideal point, that is, have a common perpendicular DE (fig. 1). Then it is easily shown that D and E lie on the side of BC away from A, for otherwise it would follow that the sum of four angles of a hyperbolic quadrilateral are together greater than four right angles or that an exterior angle of a triangle is less than the interior opposite angle.

This common perpendicular DE cannot meet BC produced either towards B or towards C for in either case an exterior angle would be less than an interior opposite angle. Neither can DE be parallel to BC either towards B or towards C for then an angle of parallelism would be greater than a right angle. Therefore DE and BC meet at an ideal point, that is, have a common perpendicular GF, where it is easy to see that G lies on BC between B and C and F lies on DE between D and E.

Produce AB to H and ED to K making BH=BG and DK=DF. Also produce AC to L and DE to M making CL=CG and EM=EF. Then HK is a common perpendicular to AB and ED and LM is a common perpendicular to AC and ED. Also HK=GF=LM.

Bisect KM at N. Then the perpendiculars NP and NQ on AB and AC are equal from the equality of the quadrilaterals NPHK and NQLM. Therefore AN is the internal bisector of the angle A. It is also evidently perpendicular to DE. Therefore BD, CE and AN have a common perpendicular and therefore meet at an ideal point.

If the two external bisectors of the angles B and C are parallel, the internal bisector of the angle A cannot meet either as then the three would pass through a common actual point. The internal bisector therefore passes between the two parallel external bisectors without meeting either and therefore must be parallel to both in the same sense.

Corollary to Theorem II :—In the triangle ABC if g be the foot of the perpendicular on BC from the point O, the actual point of concurrence of the internal bisectors of the triangle ABC and G, the foot of the perpendicular on BC from O′ the actual, ideal or improper point of concurrence of the internal bisector of the angle A and the external bisectors of the angles B and C, then $Bg=CG$.

For, $$AB-Bg=AC-Cg,$$

also $$AB+BG=AC+CG,$$

as is evident from constructions of Theorems I and II when O′ is an actual or ideal point. When O′ is an improper point similar constructions have to be made.

Theorem III :—*The three perpendiculars from the vertices of a triangle in the hyperbolic plane on the opposite sides meet at a point, actual, ideal or improper.*

Let ABC be the given triangle and AD, BE, CF the three perpendiculars from A, B, C on the opposite sides. Draw α, β, γ through A, B, C at right angles to AD, BE, CF respectively.

Case 1. Suppose β and γ meet at an actual point. Then it will be shown that α, β and α, γ will also meet at actual points.

Let G be the point of intersection of β and γ (fig. 2). Produce GC to H and GB to K making CH=GC and BK=GB. Then the join of HK will pass through A and will be perpendicular to AD.

From G, H, K draw perpendiculars GL, GM, GN, HO, HP, HQ, KR, KS, KT on the sides BC, CA and AB of the triangle ABC. Then because GC=HC and CF is the common perpendicular to HG and AB we have GN=HQ. Again from the congruent triangles GBN and KBT we get GN=KT. It follows therefore HQ=KT and similarly HP=KS. If H and K be joined, the line HK will pass through A, for otherwise it will cut BA and CA or each side produced through A in two points each of which shall be the middle point of the segment HK which is absurd. Now HO=GL=KR and from the congruent quadrilaterals HODA and KRDA, it is clear that angle DAH is equal to angle DAK; thus AD is perpendicular to HK. Hence the perpendiculars from the vertices A, B, C are the perpendicular bisectors of the sides of the triangle GHK and they therefore meet at a point. (Theorem of Bolyai).

It is important to observe that as $BC=\frac{1}{2}\,OR=OD$, we get $BD=OC=CL$.

Case 2. Suppose now that β and γ are parallel, that is, meet at an improper point (fig. 3.). If on AD, between A and D, a point A′ be taken, perpendiculars BE′ and CF′ from B and C on the sides CA′ and BA′ of the triangle A′BC will lie on the sides of BE and CF away from BC. Therefore if β' and γ' be drawn through B and C perpendiculars to BE′ and CF′, they meet at an actual point G′ and it can be proved, as in *Case 1*, that a', β' and a', γ' also meet in actual points K′ and H′ where a' is the perpendicular through A′ to A′D. If G′L′ be drawn from G′ perpendicular to BC, then because DC=BL′, as A′ moves along A′D towards A, G′ will move along L′G′ away from BC and finally when A′ coincides with A, G′ moves off to infinity so that β' and γ' coincide with β and γ. Now as BG′ and CG′ are always equal to BK′ and CH′ respectively as G goes to infinity, H′ and K′ at the same time go to infinity. Again as the theorem is true in all particular cases it is also true in the limiting case; a' which is always perpendicular to A′D will remain so when A′ moves to A, that is, when a' coincides with a. Thus a which is perpendicular to AD meets β and γ at improper points.

Case 3. Let now β and γ be non-intersecting lines, that is, let them meet at an ideal point. They will have a common perpendicular GG′ representative of that point.

We may suppose the angles ABC and ACB to be acute, for at least two of the angles of a triangle must be so. Then angles CBG and BCG′ are both acute, consequently GG′ cannot cut BC.

GG′ cannot also cut AB and AC; for supposing GG′ cuts AB and AC (fig. 4) at the points G_3 and G_2, on GB and G′C produced through B and C we can take two points H and K such that GB=BH and G′C=CK. Perpendiculars erected at H and K to HB and KC will meet BA and CA, produced if necessary at H_3, H_2 and K_3, K_2 respectively. Further these pependiculars must cut each other at a point Z. It can now be easily shown that the triangles ZH_2K_2 and ZH_3K_3 are both isosceles, so that the bisector of the angle H_2ZK_2, which is also the bisector of the angle H_3ZK_3, must be perpendicular to both the intersecting lines H_2K_2 and H_3K_3 which is absurd.

Again it is not possible that GG′ shall cut one of the sides AB and AC and be parallel to the other. Supposing that GG′ cuts AB at G_3 and is parallel to AC, if on GB produced we take as before a point H such that GB=BH and erect a perpendicular at H, this perpendicular will cut BA, produced if necessary, at a point H_3. for the triangles G_3GB and H_3HB are congruent. Consequently it will cut CA, produced if necessary, at a point H_2. Now as GB=BH and BE is the common

perpendicular to HG and AC and also GG′ is parallel to AC, the perpendicular through H to HG must also be parallel to CA. Thus this perpendicular cuts CA and is at the same time parallel to CA which is absurd.

In an exactly similar way it can be shown that it is not possible that GG′ shall cut one of the sides AB and AC and be non-intersecting to the other.

Further it is easy to see in like manner that GG′ cannot be parallel to both AB and AC.

GG′ must therefore be non-intersecting to both AB and AC.

Let G_1L, G_2M, G_3N be the common perpendiculars between the lines GG′ and BC, CA and BA respectively (fig. 5). If GB and G′C be produced through B and C to H and K making GB=BH and G′C =CK and perpendiculars HH′ and KK′ be erected at these points, these perpendiculars cannot cut either BC, CA or AB; for supposing that any of these perpendiculars cuts any of the sides BC, CA or AB, it can be shown from the properties of congruent figures that GG′ must then also meet one of the sides BC, AC or AB which we have seen is not possible.

Let H_1O, H_2P, H_3Q be the common perpendiculars between HH′ and BC, CA and AB respectively and K_1R, K_2S, K_3T be the common perpendiculars between KK′ and the same three lines respectively. Then it is easy to see from congruent figures that $H_1O=G_1L=K_1R$, $K_2S=G_2M=H_2P$ and $H_3Q=G_3N=K_3T$. If now H′K′ be the common perpendicular between HH′ and KK′, H′K′ must pass through A; for considering the two figures $APH_2H'H_3QA$ and $ASK_2K'K_3TA$ in which $H_3Q=K_3T$ and $H_2P=K_2S$, it can be shown that if H′K′ does not pass through A, it will cut AB and AC, produced if necessary through A, in two points each of which shall be the middle point of the finite segment H′K′ which is absurd. Further from the equality of the figures $H'ADOH_1$ and $K'ADRK_1$ it is clear that $\angle H'AD=\angle K'AD$ and AH′=AK′, so that AD is perpendicular to H′K′ through its middle point A.

Thus AD, BE and CF are the perpendicular bisectors of the common perpendiculars H′K′, HG and G′K between HH′, KK′; GG′, HH′ and GG′, KK′ respectively. And these will be proved to be concurrent later on. *See Theorem IV, Case II(i).*

Cor. to Theorem III:—BD=CL. This is evident from the constructions in the different cases.

Definitions :—

The *symmetric* of two given directed lines is the locus of the middle points of all lines which are equally inclined to the two lines.

Every line perpendicular to the symmetric or passing through the symmetric, when the symmetric is a point, either meets the two given lines at equal angles or have equal common perpendiculars from them.

The symmetric of two given directed lines which meet at an actual point is the internal bisector of the angle between them.

The symmetric of two given directed lines which meet at an ideal point and have consequently a common perpendicular between them is the line bisecting this common perpendicular at right angles, if the given lines are directed in the same sense with respect to this common perpendicular; but if they are directed in opposite senses from this common perpendicular the symmetric reduces to the middle point of the common perpendicular, for it is evident that every line which is equally inclined to the two given directed lines passes through the middle point of the common perpendicular and is bisected at that point.

If the two given directed lines are parallel, the symmetric is a third parallel which is equidistant from them provided the given lines are both directed in the same sense as, or opposite sense to, the direction of parallelism. Otherwise the symmetric will be defined to be the improper point to which the parallel lines converge.

With these definitions we proceed to prove the following beautiful and comprehensive theorem.

Theorem IV :—*The symmetrics of any three co-planer lines, which are not concurrent, taken two and two in any three ways such that the same line has opposite senses in the two different pairs in which it occurs, are concurrent, the concurrency being understood as follows* :—

(*a*) if the three symmetrics are straight lines, they will meet at an actual, ideal or improper point;

(*b*) if two of them be straight lines and the third a point, then the point will lie on the common perpendicular to the first two;

(*c*) if one of the symmetrics be a straight line and the other two points, then the straight line will be perpendicular to the join of the two points;

(*d*) if all the three symmetrics be points they will be collinear.

Let a, b, c represent any three coplanar lines which are not concurrent. If b and c meet at an actual point we will denote this point by α. If b and c meet at an ideal point, they have a common perpendicular. The ideal point or the common perpendicular may be

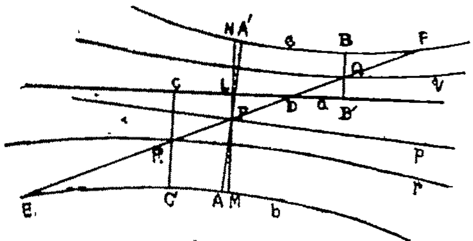

fig. 8.

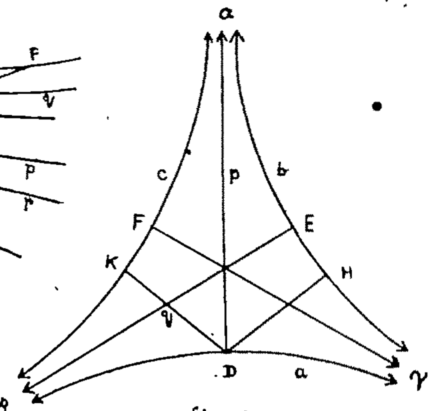

fig. 9.

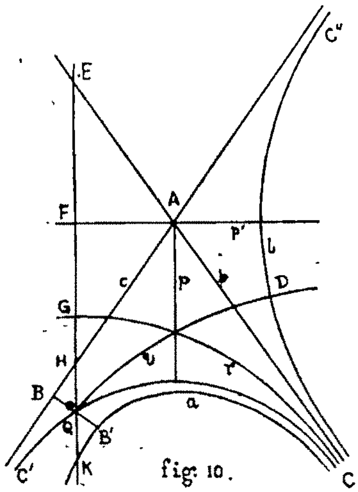

fig. 10.

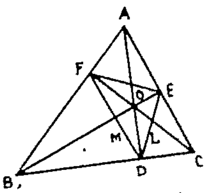

fig. 11

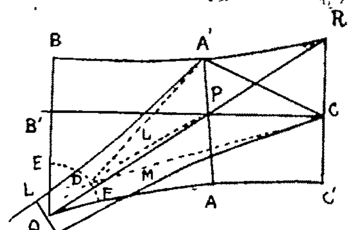

fig. 13.

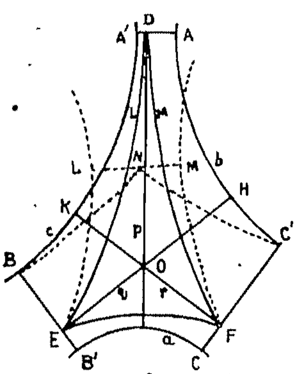

fig. 12.

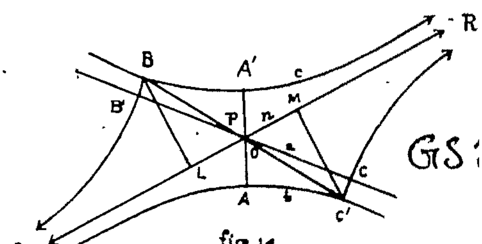

fig. 14.

Illustrating Mukhopadhyaya and Bhar's paper on generalisation of certain theorems in the hyperbolic geometry of the triangle.

indifferently denoted by α. If b and c meet at an improper point, then this improper point will be denoted by α. Similarly the points of meeting, actual, ideal or improper, of the two lines c and a will be denoted by β and that of the lines a and b by γ.

The line a is directed in two ways and may be represented as such by $\beta\gamma$ and $\gamma\beta$. If β and γ be actual points this is obvious ; if β and γ be ideal points then $\beta\gamma$ will represent lines a as the common perpendicular between β and γ directed from β towards γ. Similarly if β be an actual point and γ an ideal point, then $\beta\gamma$ will represent the line a directed from β towards γ to which it is perpendicular. Similar interpretations may be given in every case.

The three lines a, b, c can be taken in groups of directed pairs, two and two, only in four ways satisfying the condition that if any one of the lines a occur as $\beta\gamma$ in one point, it can only appear as $\gamma\beta$ in another point. These group are :

(1) $\beta\alpha,\ \gamma\alpha\ ;\ \ \gamma\beta,\ \alpha\beta\ ;\ \ \alpha\gamma,\ \beta\gamma\ ;$

(2) $\beta\alpha,\ \gamma\alpha\ ;\ \ \beta\gamma,\ \alpha\beta\ ;\ \ \alpha\gamma,\ \gamma\beta\ ;$

(3) $\beta\alpha,\ \alpha\gamma\ ;\ \ \gamma\beta,\ \alpha\beta\ ;\ \ \gamma\alpha,\ \beta\gamma\ ;$

(4) $\alpha\beta,\ \gamma\alpha\ ;\ \ \gamma\beta,\ \beta\alpha\ ;\ \ \alpha\gamma,\ \beta\gamma\ ;$

Case I :—

Let α, β, γ be actual points.

(*i*) Let the lines a, b, c be taken in directed pairs as group (1). The symmetric of $\beta\alpha$ and $\gamma\alpha$ is the internal bisector of the angle between b and c ; so the symmetrics of $\gamma\beta$, $\alpha\beta$ and $\alpha\gamma$, $\beta\gamma$ are the internal bisectors of the angles between c, a and a, b. Hence the symmetrics are concurrent (Theorem I).

(*ii*) If now the lines be taken in directed pairs as in group (2), the symmetric of $\beta\alpha$, $\gamma\alpha$ is the internal bisector of the angle between b and c ; but the symmetrics of $\beta\gamma$, $\alpha\beta$ and $\alpha\gamma$, $\gamma\beta$ are the external bisectors of the angles between c, a and a, b. So the three symmetrics are concurrent. (Theorem II).

(*iii*) If the lines be taken in directed pairs as in group (3) or (4) we have a repetition of (*ii*).

Case II :—

Let α, β, γ be ideal points and suppose every two of the lines a, b, c lie on the same side of the third. In this case no straight line can meet all the three lines at actual points.

Let AA′, BB′ and CC′ be the common perpendiculars between b, c; c, a and a, b and let P, Q, R be their middle points and p, q, r be the perpendiculars through P, Q, R to AA′, BB′ and CC′ respectively.

(*i*) When the lines a, b, c are taken in directed pairs as in group (*i*), the symmetries of $\beta\alpha$; $\gamma\alpha$; $\gamma\beta$, $\alpha\beta$ and $\alpha\gamma$, $\beta\gamma$ are p, q and r respectively.

Suppose q and r meet at an actual point O. Perpendiculars OM and ON on the sides b and c are equal being each equal to the perpendicular OL on a. Therefore the symmetric p passes through O. Thus the three symmetries p, q, r are concurrent at an actual point O.

If however q and r meet at an ideal point, they have a common perpendicular O_2O_3 (fig. 6). Now O_2O_3 cannot meet b or c; for if it meets b, it must meet a, but it is evident that it cannot meet a since if it meets a it cannot meet q. Let O′L, O″M, O‴N be the common perpendiculars between O_2O_3 and a, b, c. Then $O''M = O'L = O'''N$. Therefore the common perpendicular to O_2O_3 and AA′ bisects AA′, that is PO_1 is perpendicular to O_2O_3. Thus p, q, r have a common perpendicular, that is the three symmetrics have a common ideal point.

Lastly if q and r be parallel p is also parallel to them in the same sense. For as A and A′ are points on opposite sides of q as well as of r, the line AA′ must meet q and r at some points D and E (fig. 6). Let DF and EG be the perpendiculars from D and E on a. Then $DF < DG < DE + EG = DA$, $\therefore$ $DA' = DF < DA$. Similarly $EA = EG < EA'$. Hence P lies between D and E. Now p cannot meet q at an actual point as then r would pass through the same point and consequently could not be parallel to q; likewise p cannot meet r at an actual point. Thus r falls between the parallel lines q and r but does not meet either. Hence p must be parallel to q and r in the same sense. The three symmetrics are therefore concurrent at an improper point.

(*ii*) If now the lines be taken in directed pairs as in group (2), the symmetric of $\beta\alpha$, $\gamma\alpha$ is the line p; but the symmetrics of $\beta\gamma$, $\alpha\beta$ and $\alpha\gamma$, $\gamma\beta$ are the points Q and R. We are to show therefore that p is perpendicular to the join of Q and R.

The line QR (fig. 7) cannot meet any of the sides a, b, c; for if it meets one, it meets all the three which is impossible. If O′L, O″M, O‴N be the common perpendiculars between QR and the sides a, b, c, it is easy to see that $O''M = O'L = O'''N$, therefore the common perpendicular between AA′ and QR bisects AA′, thus p is perpendicular to QR.

(*iii*) If the lines be taken in directed pairs as in group (3) or (4), we get a repetition of (*ii*).

Case III :—

Let a, β, γ be ideal points and suppose the lines a, b, c be so related that one of them a has b and c on opposite sides.

Let AA′, BB′, CC′ be the common perpendiculars to the line pairs (b,c), (c,a) and (a,b); P, Q, R their middle points and p, q, r the perpendiculars through P, Q, R to AA′, BB′ and CC′ respectively (fig. 8).

(ii) If the lines be taken in directed pairs as in group (1), the symmetric of βa and γa is the point P, but the symmetries of $\gamma\beta$, $a\beta$ and $a\beta$, $\beta\gamma$ are the lines q and r. We are to show therefore that the common perpendicular to q and r passes through P.

Let the common perpendicular to q and r meet the sides a, b, c at L, M, N. LMN being perpendicular to q, $\sphericalangle$BNL$=\sphericalangle$B′LN. Similarly $\sphericalangle$C′ML$=\sphericalangle$CLM. But $\sphericalangle$B′LN$=\sphericalangle$CLM. $\therefore$ $\sphericalangle$BNL$=\sphericalangle$C′ML. Therefore LMN passes through the middle point of AA′, that is, through the point P.

(ii) The lines being taken in directed pairs as in group (2), the three symmetrics are the points P, Q, R We are to show therefore in this case that P, Q, R are collinear.

Let the line QR meet the three sides a, b, c in the points D, E, F. Then $\sphericalangle$BFQ$=\sphericalangle$B′DQ$=\sphericalangle$ODR$=\sphericalangle$C′ER. $\therefore$ EDF bisects AA′, that is P lies on QR.

(iii) If the lines be taken in directed pairs as in group (3) or (4) we get a repetition of (i).

Case IV :—

Let a, β, γ be each an improper point.

Let p be the parallel to b and c which is equidistant from both, q the parallel to c and a equidistant from c and a and r the parallel to a and b equidistant from a and b (fig. 9).

(i) Suppose the lines are taken in directed pairs as in group (1). The symmetrics of βa, γa; $\gamma\beta$, $a\beta$ and $a\gamma$, $\beta\gamma$ are p, q, r respectively. We are to show that p, q, r are concurrent.

As q is parallel to c and a it cannot cut either again; it must therefore meet b at an actual point E and r must therefore cut q at an actual point O. The perpendiculars OM and ON on b and c are equal, each being equal to the perpendicular OL on a. Therefore O lies on p. Hence the three symmetrics pass through the same point O.

(*ii*) If the lines be taken in directed pairs as in group (2), the symmetric of βa, γa is the line p but the symmetrics of $\beta\gamma$, $a\beta$ and $a\gamma$, $\gamma\beta$ are the improper points β and γ so that their join is the line a. We are to show that p is at right angles to a.

The line p must meet a at an actual point D. The perpendiculars DH and DK on b and c are equal; hence $\measuredangle a$DH and $\measuredangle a$DK are equal being angles of parallelism for equal distances DH and DK. For a like reason, $\measuredangle\beta$DK$=\measuredangle\gamma$DH. Therefore p is at right angles to a.

(*iii*) The other arrangements of the directed pairs will given repititions of (*ii*).

Case V :—

Let a be an actual point, β an ideal and γ an improper point.

Let A represent the actual point a, BB′ the common perpendicular between c and a represent the ideal point β and C represent the improper point γ, let p and p' be the internal and external bisector of the angles BAC, q the perpendicular to BB′ through its middle point Q and r the parallel to a and b which is equidistant from them (fig. 10).

(*i*) If the arrangement of the directed pairs be as in group (1) the three symmetries are p, q, r and the proof of their concurrency proceeds on the lines of the proof for this arrangement in the other cases.

(*ii*) Let the arrangement of the directed lines be as in group (2), The symmetric of βa, γa is p; but the symmetries of $\beta\gamma$, $a\beta$ and $a\gamma$, $\gamma\beta$ are the points Q and C. We are to show that p is perpendicular to QC which is the parallel through Q to a and b.

If QC be produced through Q to C′ on the side of BB′ away from C, QC′ will be parallel to AB in the direction AB as Q is the middle point of BB′. Hence p is perpendicular to QC [case IV (*ii*)].

(*iii*) If the lines be taken in directed pairs as in group (3), the symmetric of βa, $a\gamma$ is p', that of $\gamma\beta$, $a\beta$ is q which that of γa, $\beta\gamma$ is the point C. We are to prove therefore that the common perpendicular to p' and q passes through the improper point C, that is, is parallel to a and b.

Through Q draw QC parallel to a and b and on q cut off the length QD corresponding the angle of parallelism DQC. The line l through D perpendicular to q will be parallel to a and b in the same sense and when produced through D will meet BA produced at an improper point

C. p' will therefore be perpendicular to l [case IV (*ii*)]. Hence the common perpendicular to p' and q is parallel to a and b.

(*iv*) Lastly let the lines be taken in directed pairs as in group (4); the symmetric of $\alpha\beta$, $\gamma\alpha$ is the line p', that of $\gamma\beta$, $\beta\alpha$ is the point Q, whereas of $\alpha\gamma$, $\beta\gamma$ is the line r. We are to show that the common perpendicular to p' and r passes through Q.

Let the perpendicular through Q to r meet the lines b, p', r, c and a at the points E, F, G, H and K respectively, then $\sphericalangle QKB' = \sphericalangle QHB$ and $\sphericalangle QKB' = \sphericalangle QEA$; $\therefore AEH = \sphericalangle AHE$. Hence p is perpendicular to QE.

If two of the points α, β, γ be actual and the third ideal or improper, or two of them ideal and the third actual or improper, or finally, if two of them be improper and the third actual or ideal, the proof of the theorem proceeds on similar lines and requires no further special investigation.

Moreover when one at least of the three points α, β, γ is an ideal one, the triangle formed by the three lines a, b, c may either be such that any two of the three lines a, b, c lie on the same side of the third, or that one of the lines a, b, c has the other two on opposite sides of it but as the proof of theorem in all these varied cases requires no new principles, the different cases are not separately discussed.

We shall now extend the theorem of the perpendiculars (Theorem III) to the case of a triangle of which the vertices may be actual, ideal or improper.

Theorem V. In a triangle formed by any three co-planar lines a, b, c intersecting not necessarily in actual points the perpendiculars p, q, r, from the vertices α, β, γ—actual, ideal or improper—on the opposite sides are concurrent; the concurrency being interpreted as in Theorem IV.

When α is an ideal point the perpendicular p on a is the common perpendicular between a and the line α if a and α be non-intersecting; when a and α meet at an actual or improper point, the common perpendicular reduces to this actual or improper point. If α be an improper point, the perpendicular p is the perpendicular to a which is parallel to b and c in the same sense.

Case I:—

Let α, β, γ be actual points.

The theorem for this case has been already proved (Theorem III). The following is a more general proof and is applicable to almost all the cases.

Let A, B, C represent the three actual points α, β, γ and suppose the perpendiculars BE and CF on AC and AB meet at O (fig. 11). Draw the lines EL and F M making $\measuredangle$BEF$=\measuredangle$BEL and $\measuredangle$CFE$=\measuredangle$CFM. Then BE, AC and CF, AB are the internal and external bisectors of the angles FEL and EFM. It follows from the theorem of symmetrics (Theorem IV) that as AC and FC meet at an actual point C, EL and FM must also meet at an actual point D and OD is the internal bisector of the angle EDF. Again from the same theorem it follows that the line through D perpendicular to OD must pass through B and C; in other words D must lie on BC and OD is perpendicular to BC. But OD passes through A as A is the point of intersection of the external bisectors of the angles E and F. Hence the perpendicular from A on BC passes through O. The three perpendiculars are therefore concurrent. Similar proof holds if the point O be ideal or improper.

Case II.:—

Let α, β, γ be ideal points and suppose every two of the lines a, b, c be on the same side of the third (fig. 12).

Let BB′ and CC′ be the common perpendiculars between c, a and a, b representing the points β and γ and EH and FK the common perpendiculars q and r between the lines β, b and γ, c and let q and r meet at the point O. Join EF and draw EL and FM making $\measuredangle$HEL$=\measuredangle$HEF and $\measuredangle$KFM$=\measuredangle$KFE. Suppose EL and FM meet at the actual point D. Then DO bisects $\measuredangle$EDF and it follows from Theorem IV that OD is the perpendicular to a. If now ADA′ be drawn at right angles to OD, from the same theorem it follows that ADA′ is perpendicular to both b and c, so that ADA′ is the common perpendicular between b and c. Hence OD is the common perpendicular between a and α. The three perpendiculars p, q, r are therefore concurrent.

If EL and FM do not meet at an actual point, they must be either non-intersecting or parallel. Suppose in the first place that they are non-intersecting. Let LM represent the common perpendicular between them and N be the middle-point of LM. From Theorem IV it follows that the perpendicular between EH and CC′ must pass through N, Similarly c must pass through N, Thus b and c meet at an actual point N which is contrary to hypothesis. Hence EL and FM cannot be non-intersecting. If now EL and FM are parallel, it can be shown in like manner, that b and c must be parallel to them in the same sense which

is against hypothesis. Hence EL and FM must meet at an actual point D.

Case III:—

Let α, β, γ be ideal points and the lines a, b, c be so related that two of them, b and c, lie on opposite sides of the third line a. Then the common perpendicular AA′ between b and c must meet a at some point P.

(*i*) Suppose the common perpendiculars BB′ and CC′ between c, a and a, b meet b and c respectively at the points Q and R (fig. 13). We are to show that P, Q and R are collinear.

Join A′C. Draw the line A′L on the side of AA′ away from C making $\sphericalangle$AA′L$=\sphericalangle$AA′C. Similarly draw the line CM on the side of CB′ remote from A′ making $\sphericalangle$B′CM$=\sphericalangle$B′CA′. Then AA′, A′B and B′C, CC′ are the internal and external bisectors of the angles LA′C and MCA′. If possible let A′L and CM meet at D. The line DP and the perpendicular EDF through D to DP are the internal and external bisectors of the angle ADC. From Theorem IV it follows that EDF is perpendicular to both the intersecting lines BB′ and C′A which is absurd. Hence A′L and CM cannot meet at an actual point. Nor can they be parallel, for from the same theorem it would follow that BB′ and C′A would be parallel each being parallel to the lines A′L and CM in the same sense and this is impossible. Hence A′L and CM are non-intersecting.

From the same theorem it is further evident that Q is the middle point of the common perpendicular LM between A′L and CM. Hence the line through Q perpendicular to LM must pass through p, the point of intersection of the internal bisectors of the angles CA′L and A′CM and also through R, the point of intersection of the external bisectors of the same angles, Thus P, Q, R are collinear.

(*ii*) Let BB′, b and CC′, c be non-intersecting lines and l, m, be the common perpendiculars between them. We are to show that the common perpendicular to l and m must pass through P.

If we consider the ideal triangle of which the sides are b, m and BB′, the theorem follows from Case II.

(*iii*) Let now BB′, b and C′C, c be pairs of parallel lines (fig. 14), and n be a line which is parallel to both BB′ and c and therefore to b and C′C in the same senses. We are to show that P lies on n.

Let BL and C′M be the perpendiculars from B and C′ on the line n. It is evident BL$=$C′M being the lengths corresponding to angles of

parallelism each equal to half of a right angle. Hence BC′ is bisected at the point O where it meets LM. Therefore the perpendicular through O to b will also be perpendicular to c; similarly the common perpendicular to BB′ and CC′ will pass through O and hence O coincides with P. Thus P lies on n.

Case IV :—

When α, β, γ are improper points, the three symmetries p, q, r are the perpendiculars from α, β, γ on a, b, c respectively. [Theorem IV, Case IV, (*ii*) (*iii*), (*iv*)]. Hence the three perpendiculars are concurrent [Theorem IV Case IV (*i*)].

5

A NOTE ON QUIN-QUI-SECTION

By

Pandit Oudh Upadhyaya.

[*Read January 16th, 1921.*]

The problem of quin-qui-section has been completely solved by S. W. Burnside in the *Proceedings of the London Mathematical Society*, 1915. There he has shown that this problem depends on the solution of two Diophantine equations which are as follows :—

(1) $[4p-16-25(A+B)]^2+1125(A-B)^2+450(C^2+D^2)=12^2p$,

(2) $[4p-16-25(A+B)][A-B]+3(C^2+4CD-D^2)=0$.

He has solved these equations for the primes 11, 31, 41, 61 and 71 and has given the values of p, A, B, C and D in a tabular form.

The object of this short note is to solve the same problem for the primes 1511, 1531, 1571 and 1601. I have also calculated the quintic in question with the help of the following formula given by W. Burnside in the same paper :—

$$\eta^5+\eta^4-\frac{2(p-1)}{5}\eta^3+\left[\frac{p(A+B)}{3}-\frac{2(p-1)(2p+3)}{3\times5\times5}\right]\eta^2$$

$$+\left[\frac{p\left(\frac{p-1}{5}+A+B\right)^2}{9}-pAB-\left(\frac{p-1}{5}\right)^3\right]\eta$$

$$+\frac{p}{5}\left[\frac{1}{5\cdot6^3}\left\{5(A+B)-\frac{4p-4}{5}\right\}^3+\frac{1}{6^2}\left\{\frac{2p-2}{5}-A-B\right\}^2\right.$$

$$\left.+\frac{(A-B)^2}{4}+\frac{(A-B)(D^2-C^2)}{8}\right]-\frac{(p-1)^3}{5^4}=0$$

In calculating the quintic, I have used the above formula and this formula is the same as that given by W. Burnside except that in the co-efficient of η I have used $\left(\frac{p-1}{5}\right)^3$ instead of $\frac{(p-1)^3}{5^3}$, which, I think, is a misprint in his paper.

There seems to be another misprit on page 257 in the last line. He has written $| 14p-16-25(A+B) | +15\sqrt{5}(A-A)<12\sqrt{p}$

Instead of 14, it ought to be 4.

(1) Calculation for the prime 1511.

$$[4p-16-25(A+B)]^2+1125(A-B)^2+450(C^2+D^2)=12^2p \quad \dots \text{(1)}$$

$$[4p-16-25(A+B)][A-B]+3(C^2+4CD-D^2)=0. \quad \dots \text{(2)}$$

In the first equation let us substitute the value of p, supposing that $A+B=253$, then we get by the first equation

$$[4\times1511-16-25\times253]^2+1125(A-B)^2+450(C^2+D^2)=144\times1511,$$

or $$1125(A-B)^2+450(C^2+D^2)=217584-88209.$$

Now if $A-B=5$,

$$1125\times5^2+450(C^2+D^2)=129375.$$

$$\therefore C=12 \text{ and } D=9.$$

And because $A+B=253$ and $A-B=5$,

$$\therefore A=129 \text{ and } B=124.$$

Substituting these values in the second equation we get

$$[4\times1511-16-25\times253][5]+3(12^2+4\times12\times9-9^2)=0.$$

Thus it is clear that these values ($A=129$, $B=124$, $C=12$, and $D=9$) satisfy both the equations. Now,

the co-efficient of $\eta^3=-\dfrac{2(p-1)}{5}=-604$,

the co-efficient of $\eta^2=\dfrac{p(A+B)}{3}-\dfrac{2(p-1)(2p+3)}{3\times5\times6}=5621.$

the co-efficient of $\eta=\dfrac{p}{9}\left(\dfrac{p-1}{5}+A+B\right)^2-pAB-\left(\dfrac{p-1}{5}\right)^2=411.$

The constant term $=\dfrac{p}{5}\left[\dfrac{1}{5\cdot6^3}\left\{5(A+B)-\dfrac{4p-4}{5}\right\}^3\right.$

$\left.+\dfrac{1}{6^2}\left(\dfrac{2p-2}{5}-A-B\right)^2+\dfrac{1}{4}(A-B)^2+\dfrac{1}{8}(A-A)(D^2-C^2)\right]$

$$-\frac{(p-1)^3}{5^5}=-25731.$$

The quintic is therefore

$$\eta^5-\eta^4-604\,\eta^3+5621\,\eta^2-411\,\eta-25731=0.$$

(2) *The calculation for the prime 1531.*

$$[4p-16-25(A+B)]^2+1125(A-B)^2+405(C^2+D^2)=12^2p \;\ldots\; (1)$$

$$[4p-16-25(A+B)][A-B]+3(C^2-4CD-D^2)=0 \quad\ldots\; (2)$$

In the first equation let us substitute the value of p supposing $A+B=261$, then we get by the first equation :

$$[4\times1431-16-25\times261]^2+1125(A-B)^2+450(C^2+D^2=144\times1531,$$

or $$1125(A-B)^2+450(C^2+D^2)=46575.$$

Now if $A-B=1$,

$$1125(1)^2-450(C^2+D^2)=46575.$$

$$\therefore\; C=10 \text{ and } D=1.$$

Also because $A+B=261$ and $A-B=1$,

$$\therefore\; A=131 \text{ and } B=130.$$

Substituting these values in the second equation we get :

$$[4\times1531-16-25\times261][1]+3(10^2+4\times10\times1-1^2).=0.$$

Thus it is evident that these values (A=131, B=130, C=10 and D=1) satisfy both the equations

The quintic in this case is

$$\eta^5+\eta^4-612\,\eta^3+8145\,\eta^2-36595\,\eta+48741=0.$$

(3) *Calculation for the prime 1571.*

$$[4p-16-25(A+B)]^2+1125(A-B)^2+450(C^2-D^2)=12^2p \;\ldots\; (1)$$

$$[4p-16-25(A+B)][A-B]+3(C^2+4CD-D^2=0 \quad\ldots\; (2)$$

Let us suppose that $A+B=247$, then substituting the values of p and $A+B$ in the first equation, we get

$$(6284-16-6176)^2+1125(A-B)^2+450(C^2+D^2)=226224$$

or $$1125(A-B)^2+450(C^2+D^2)=217575$$

If $$A-B=-11,\; C^2+D^2=181=10^2+9^2.$$

$$\therefore\; C=9,\; D=10.$$

When $A+B=247$ and $A-B=-11$, then it evident that $A=118$, $B=129$.

Hence $A=118$, $B=129$, $C=9$ and $D=10$ is the required solution as will be evident by substituting these values in the second equation.

Substituting these values in the second equation, we get,

$$[4\times1571-16-25\times247]\times[-11]+3(9^2+9\times10\times4-10^2)=0.$$

This it is clear that these values satisfy both the equations.
The required quintic is $\eta^5+\eta^4-628\,\eta^3-2325\,\eta^2+63393\,\eta+27169=0$.

(4) *Calculation for the prime 1601.*

$$[4p-16-25(A+B)]^2+1125(A-B)^2+450(C^2+D^2)=12^2p \quad \ldots \quad (1)$$

$$[4p-16-25(A+B)][A-B]+3(C^2+4CD-D^2)=0 \quad \ldots \quad (1)$$

In the first equation let us substitute the value of p supposing that $A+B=265$, then we get by the first equation:

$$[4\times1601-16-25\times265]^2+1125(A-B)^2+450(C^2+D^2)=144\times1601.$$

Now if $\qquad A-B=5$

$$1125(5)^2+450(C^2+D)=174375.$$

$$\therefore\ C=18 \text{ and } D=1.$$

Also because $\qquad A+B=265$ and $A-B=5$.

$$\therefore\ A=135 \text{ and } B=130.$$

Substituting these values in the second equation we get:

$$[4\times1601-25\times265-16][5]+3(18^2+4\times18\times1-1^2)=0.$$

Thus it is clear that these values, ($A=135$, $B=130$, $C=18$ and $D=1$), satisfy both the equations.

Hence the quintic is

$$\eta^5+\eta^4-640\,\eta^3+4675\,\eta^2+12475\,\eta-95561=0.$$

6

Cyclotomic Hepta-section

By

Pandit Oudh Upadhyaya

[*Read January 16th, 1921*]

The problem of cyclotomic section has engaged the attention of many eminent mathematicians and solution has been obtained by them for particular cases.

The problem of tri-section and quarti-section was solved by A. Cayley. He also attempted the problem of quin-qui-section but could not complete the solution. Once again, he attempted the same problem in a paper published in the Proceedings of the London Mathematical Society for 1881. The solution of the problem of quin-qui-section has recently be encompleted by W. Burnside in the Proceedings of the London Mathematical Society for 1915 and this paper comes to an end with the following remarks: "I have carried the case $q=7$ so far as to assure myself that it is not quite parallel with that of $q=5$. A set of three simultaneous Diophantine relations occur, but they are not sufficient to ensure that the equations expressing the products of A's form a consistent multiplication table."

In view of the last fact, mentioned by W. Burnside it is believed that the case of hepta-section, even for particular cases, has not been considered by any other previous writer. The object of this paper is to consider the problem of hepta-section for the prime 29.

Let a be a special root of

$$x^{29}-1=0,$$

then

$$a=1 \text{ or } 1+a+a^2+a^3+a^4+\dots a^{28}=0\,;$$

Let all the special roots be divided into seven groups by the following scheme :—

$$A=a+a^{28}+a^{12}+a^{17},$$

$$B=a^{2}+a^{27}+a^{5}+a^{21},$$

$$C=a^{25}+a^{10}+a^{19}+a^{4},$$

$$D=a^{9}+a^{21}+a^{8}+a^{20},$$

$$E=a^{16}+a^{13}+a^{18}+a^{11},$$

$$F==a^{3}+a^{26}+a^{7}+a^{22},$$

and

$$G=a^{6}+a^{23}+a^{15}+a^{11},$$

In all the calculations worked out below 1 has been substituted for a^{29}.

It is evident that

$$A+B+C+D+E+F+G=(1+a+a^{2}+a^{3}+a^{4}+\dots+a^{28})=-1.$$

Now all the roots have been divided into seven groups namely A, B, C, D, E, F and G, and the number of all multiplications of these groups taken two at a time is 21.

The value of each multiplication is

$$AB=a^{3}+a+a^{14}+a^{19}+a^{28}+a^{26}+a^{10}+a^{15}+a^{25}+a^{23}+a^{7}+a^{12}+a^{6}+a^{4}+a^{17}+a^{22},$$

$$AC=a^{5}+a^{3}+a^{16}+a^{21}+a^{26}+a^{24}+a^{8}+a^{13}+a^{20}+a^{18}+a^{2}+a^{7}+a^{11}+a^{9}+a^{22}+a^{27},$$

etc.

Summing, we get

$$\Sigma AB=12(1+a^{2}+a^{3}+a^{4}+a^{5}+a^{6}\dots+a^{28})=-12.$$

The number of multiplications taken three at a time is 35. The value of each multiplication is as follows :—

$$\text{ABC}=4+a+2a^2+3a^3+2a^4+2a^5+2a^6+3a^7+2a^8+2a^9+2a^{10}$$
$$+3a^{11}+a^{12}+3a^{13}+2a^{14}+2a^{15}+3a^{16}+a^{17}+3a^{18}+2a^{19}$$
$$-2a^{20}+2a^{21}+3a^{22}+2a^{23}+2a^{24}+2a^{25}+3a^{26}+2a^{27}+a^{28},$$

$$\text{ABD}=2a+3a^2+2a^3+2a^4+3a^5+3a^6+2a^7+2a^8+2a^9+2a^{10}2a^{11}2a^{12}$$
$$+2a^{13}+3a^{11}+3a^{15}+2a^{16}+2a^{17}+2a^{18}+2a^{19}+2a^{20}+2a^{21}$$
$$+2a^{22}+3a^{23}+3a^{24}+2a^{25}+2a^{26}+3a^{27}+2a^{28}.$$

etc.

Similarly other expressions can be calculated.

By actual multiplication it is found that

$$\Sigma\text{ABC}=84+77(1+a+a^2+a^3+a^4+\ldots a^{28})=7.$$

There will be 35 multiplications taken four at a time.

By actual multiplication, we can very easily find the value of each multiplication and then by addition, we get

$$\Sigma\text{ABCD}=308+309(1+a+a^2+a^3+a^4+a^5+\ldots a^{28})=-1.$$

Similarly

$$\Sigma\text{ABCDE}=728-742(a+a^2+a^3+a^4+a^5+\ldots a^{28})=+4,$$

and

$$\Sigma\text{ABCDEF}=980+989(1+a+a^2+a^3+a^4+a^5+\ldots a^{28})=-9,$$

and

$$\text{ABCDEFG}=564+565(1+a+a^2+a^3+a^4+a^5+\ldots a^{28})=-1.$$

Now with the help of the theorem, given in Art 23 of the Theory of Equations, by W. Burnside and A. W. Panton Vol. I. the required equation is found to be

$$x^7+x^6-12x^5-7x^4-x^3+14x^2-9x+1=0. \qquad \ldots \quad \text{(A)}$$

If p is a prime number and q a factor of $p-1$, there is an equation of degree q with rational co-efficients, each of whose roots is the sum of $\frac{p-1}{q}$ of the primitive pth roots of unity; no such pth root occurring in more than one of the sums. Therefore the equation is an Abelian equation with cyclical group. [See Mathew's Theory of Numbers or Smith's Report on the Theory of Numbers]. Accordingly the equation is an Abelian equation and hence it can be solved algebraically. [See Netto's Theory of Substitution].

7

Binary Commutative Algebras

By

R. Vaidyanath Swami

Linear Associative Algebras have been extensively studied, both for their formal interest and their many and wide applications, specially to the theory of Linear Groups and the matric algebra. But linear algebras without the associative property have no such applications and are for more difficult to investigate; and this remains true for the most part even if the commutative property be assumed. It would seem in fact that the associative property is a more stringent restriction on an algebra than the commutative. Nevertheless the commutative algebras have a certain claim to importance as they supply a symbolic calculus for the study of a class of rational geometric transformations. For instance a theory of transformations of the type $x' : y' = ax^2 + bxy + cy^2 : a'x^2 + b'xy + c'y^2$ may be modelled upon the lines of the theory of binary commutative algebras.

Probably the most striking of the various types of binary commutative algebras is what I have termed the 'Normal Algebra'—in which a sort of quasi-associative relation holds between any four elements. This algebra symbolises a type of relations widely prevalent in one—dimensional geometric fields—the characteristic example being the nodal cubic. The bearing of the algebra—in its multiplicative aspect—on the properties of the nodal cubic was dealt with in two papers in the Journal of the Indian Mathematical Society, Vol. IX and X.

I. Consider a linear algebra over the field of complex numbers defined by the two units e_1, e_2. A commutative multiplicative operation may be defined for this algebra by assuming arbitrary equations of the form.

$$e_1^2 = \lambda_1 e_1 + \lambda_2 e_2,$$

$$e_1 e_2 = e_2 e_1 = \mu_1 e_1 + \mu_2 e_2,$$

$$e_2^2 = \nu_1 e_1 + \nu_2 e_2.$$

The commutative algebra itself may be denoted by the matrix

$$\begin{vmatrix} \lambda_1 & \lambda_2 \\ \mu_1 & \mu_2 \\ \nu_1 & \nu_2 \end{vmatrix}$$

Ex. (1). $f_r(x) = \lambda_r x_1^2 + 2\mu_r x_1 x_2 + \nu_r x_2^2 \quad (r=1, 2.)$

Shew that $x^2 = (x_1 e_1 + x_2 e_2)^2 = f_1(x)\, e_1 + f_2(x)\, e_2$.

As in the general theory of Linear Algebras (Cf. Dickson, Linear Algebras) we find that the characteristic determinant of the general number $x (= x_1 e_1 + x_2 e_2)$ of the algebra is

$$\begin{vmatrix} \lambda_1 x_1 + \mu_1 x_2 - \omega & \lambda_2 x_1 + \mu_2 x_2 \\ \mu_1 x_1 + \nu_1 x_2 & \mu_2 x_1 + \nu_2 x_2 - \omega \end{vmatrix} = \omega^2 - \omega L(x) + D(x),$$

where $\qquad L(x) = (\lambda_1 + \mu_2) x_1 + (\mu_1 + \nu_2) x_2,$

$$D(x) = (\lambda_1 \mu_2 - \lambda_2 \mu_1) x_1^2 + (\lambda_1 \nu_2 - \lambda_2 \nu_1) x_1 x_2 + (\mu_1 \nu_2 - \mu_2 \nu_1) x_2^2.$$

$D(x)$ is the determinant of the number x and one may notice that it is also the quadratic apolar to $f_1(x)$ and $f_2(x)$.

The condition that there exists a number y to 0 such that $xy=0$ is seen to be $D(x)=0$. Hence if x, y be two *incongruent* solutions of $D(x) = 0$ (*i.e.* two solutions not merely scalar multiples of one another) it follows that $xy=0$. x and y will be called *nil-factors.*

$L(x)$ and $D(x)$ are absolute covariants of the algebra for linear transformation of units (Dickson). Also the identical relation satisfied by any number x is

$$x^3 - x^2 L(x) + x D(x) = 0, \quad \text{(Dickson).}$$

Thus a binary commutative algebra is in general of rank 3.

Ex. (2). Prove that $D(x, y) = \frac{1}{2}\left(y_1 \frac{d}{dx_1} + y_2 \frac{d}{dx_2} \right) D(x)$ is an absolute covariant of any two numbers x, y of the algebra. Establish the identities

$$(1) \quad D(x+y) \equiv D(x) + 2D(x, y) + D(y)$$

$$(2) \quad D(y)\, x^2 - 2D(x, y)\, xy + D(x)\, y^2 \equiv 0.$$

Ex. (3). If $x_1 x_2 x_3$ $y_1 y_2 y_3$ are six numbers of the algebra, then

$$\begin{vmatrix} D(x_1, y_1) & D(x_1, y_2) & D(x_1, y_3) \\ D(x_2, y_1) & D(x_2, y_2) & D(x_2, y_3) \\ D(x_3, y_1) & D(x_3, y_2) & D(x_3, y_3) \end{vmatrix} \equiv 0.$$

Ex (4). By differentiating the identical equation of the algebra, *viz.*,

$$x^3 - x^2 \, L(x) + x \, D(x) = 0,$$

show that $\Sigma xyz - \Sigma L(x) \; yz + \Sigma D(y, z) \; x \equiv 0$, where xyz are any three numbers of the algebra.

Ex. (5). Deduce from the last exercise that

$$(x^2)^2 - x^2(L(x^2) + D(x)) + 2x \, D(x^2, x) = 0$$

Hence the two numbers each of which is congruent with the square of the other satisfy the equation $L(x^2) + D(x) = 8$.

Ex. (6). If p, q are the nilfactors of the algebra, q is congruent to $p^2 - pL(p)$.

If a, b are scalars $(ap + bq)^2 = (ap - bq)^2$. Hence any number of the algebra has four square roots which are of the form $\pm ap \pm bq$. If α, β are two incongruent square roots $D(\alpha, \beta) = 0$ and $D(\alpha) + D(\beta) = 0$. Generally a number has n^2 n^{th} roots of which only n are mutually incongruent.

II. *Transformation of Units.*

Write $\begin{matrix} e_1 = p_1 e'_1 + p_2 e'_2 \\ e_2 = q_1 e'_1 + q_2 e'_2 \end{matrix}$, M = the matrix $\begin{vmatrix} p_1 & p_2 \\ q_1 & q_2 \end{vmatrix}$, m = determinant of M, M′ = the matrix $\begin{vmatrix} p_1^2 & 2p_1 p_2 & p_2^2 \\ p_1 q_1 & p_1 q_2 + p_2 q_1 & p_2 q_2 \\ q_1^2 & 2q_1 q_2 & q_2^2 \end{vmatrix}$, p's and q's being scalars.

We have from the multiplication table of the algebra,

$$\begin{vmatrix} e_1^2 \\ e_1 e_2 \\ e_2^2 \end{vmatrix} = \begin{vmatrix} \lambda_1 & \lambda_2 \\ \mu_1 & \mu_2 \\ \nu_1 & \nu_2 \end{vmatrix} \begin{vmatrix} e_1 \\ e_2 \end{vmatrix} = \begin{vmatrix} \lambda_1 & \lambda_2 \\ \mu_1 & \mu_2 \\ \nu_1 & \nu_2 \end{vmatrix} M \begin{vmatrix} e'_1 \\ e'_2 \end{vmatrix}.$$

But $e_1^2 = p_1^2 e'^2_1 + 2p_1 p_2 e'_1 e'_2 + p_2^2 e'^2_1$, etc.

Hence $\begin{vmatrix} e_1^2 \\ e_1 e_2 \\ e_2^2 \end{vmatrix} = M' \begin{vmatrix} e'^2_1 \\ e'_1 e'_1 \\ e'^2_1 \end{vmatrix}$ so that $\begin{vmatrix} e'^2_1 \\ e'_1 e'_2 \\ e'^2_2 \end{vmatrix} = M'^{-1} \begin{vmatrix} \lambda_1 & \lambda_2 \\ \mu_1 & \mu_2 \\ \nu_1 & \nu_2 \end{vmatrix} M \begin{vmatrix} e'_1 \\ e'_2 \end{vmatrix}$.

Thus if T is the matrix of the original algebra with units $e_1 e_2$, the matrix of the transformed algebra with units $e'_1 e'_2$ would be M'^{-1} TM. It is not difficult to verify from this formula the absolute covariance of $L(x)$ and $D(x)$.

The complete irreducible system of covariants of the algebra.

We must now introduce a very important covariant f of the algebra defined by

$$f(x)=x_1 f_2(x)-x_2 f_1(x) \equiv \lambda_2 x_1^3+(2\mu_2-\lambda_1)x_1^2 x_2 +(\nu_2-2\mu_1)x_1 x_2^2-\nu_1 x_2^3.$$

That $f(x)$ is a covariant of the algebra might be inferred directly from the fact that x^2 is congruent with x whenever $f(x)=0$; or it might be verified directly from the formula for transformation of units that $f\ (x', \lambda', \mu', \nu')=mf\ (x, \lambda, \mu, \nu)$. Since $f(x)$ and $L(x)$ are both linear in the constants of the algebra and since they together possess six co-efficients, it follows that the covariants f and L being known the algebra is determined uniquely. We proceed to shew that every covariant of the algebra is a covariant of f and L *regarded as binary forms.*

Theorem. If $\phi(x)$ be a covariant of the binary algebra A, then any covariant of $\phi(x)$ regarded as a binary form in $x_1\ x_2$ is also a covariant of A.

Proof. Let $\phi\ (x)$ be a covariant of weight t of A and $\psi(x)$ be a covariant of weight w of the binary form ϕ, degree r in the co-efficients of ϕ. Let c stand for the constants of the algebra.

We have $\phi(x', c')=m^t \phi\ (x, c)=m^t \phi_1\ (x', c)$ (say).

Now when the units are transformed by a matrix of determinant m, x_1, x_2 would be transformed by the conjugate inverse matrix of determinant m^{-1}. By this transformation of $x_1 x_2$, $\phi(x, c)$ is transformed into $\phi_1(x', c)=m^{-t}\phi(x', c')$. The new function ψ corresponding to the transformed form is therefore $m^{-rt}\psi\ (x', c')$. Since ψ is a covariant of weight w we must have

$$m^{-rt}\psi\ (x', c')=m^{-w}\psi(x, c).$$

Thus ψ is a covariant of A of weight $rt-w$.

More generally, if ϕ_1, ϕ_2,...are covariants of weights t_1, t_2 ..of A and ψ is a covariant of the binary forms ϕ_1, ϕ_2,.. of weight w and degree r_k in the co-efficients ϕ_k, then ψ is a covariant of A of weight $\Sigma rt-w$.

Ex. (7). The hessian of f, the cubicovariant of f and the Jacobian of f and L are absolute covariants of the algebra.

For the hessian $r=2,\ t=1,\ w=2$,

For the cubicovriant $r=3,\ t=1,\ w=3$,

For the Jacobian $r=r_2=1,\ t_1=1,\ t_2=0,\ w=1$.

Thus every covariant of f and L is a covariant of the algebra. Since f and L determine the algebra completely, it follows conversely that every covariant of the algebra is a covariant of f and L. The irreducible system of covariants of the algebra is therefore the same as the irreducible system of f and L and consists of the following 13 forms[1] (Vide Grace and Young, Algebra of Invariants, page 160).

Weight 0 (1) L (2) $\mathrm{H}=(f, f)^2$ (3) $\mathrm{J}=(f, \mathrm{L})$ (4) $f'=(f, \mathrm{H})$

Weight 1 (5) f

Weight -1 (6) $\mathrm{J}'=(f', \mathrm{L})$ (7) $\mathrm{L}_1=(f, \mathrm{L}^2)^2$ (8) $\mathrm{L}_2=(\mathrm{H}, \mathrm{L})$

Weight -2 (9) $\mathrm{L}'_1=(f', \mathrm{L}^2)^2$ (10) $\Delta=(\mathrm{H}, \mathrm{H})^2$ (11) $\Delta_1=(\mathrm{H}, \mathrm{L}^2)^2$ (12) $\Delta_2=(f, \mathrm{L}^3)^3$

Weight -3 (13) $\Delta_3=(f', \mathrm{L}^3)^3$.

It will be noticed that the ratios $\Delta : \Delta_1 : \Delta_2$ are absolute invariants of the algebra.

Ex. (8) Shew that $\mathrm{D}(x)=\frac{2}{9}\mathrm{L}^2+\frac{1}{2}\mathrm{H}-\frac{1}{3}\mathrm{J}$

$$\mathrm{L}(x^2)=\tfrac{2}{3}\mathrm{L}^2+\mathrm{J}$$

$$\mathrm{D}(x^2, x)=\tfrac{4}{27}\mathrm{L}^3+\tfrac{1}{2}f'+\tfrac{1}{2}\mathrm{HL}.$$

Ex. (9). Shew that $(\mathrm{J}, \mathrm{J})^2=\Delta_1$, $(\mathrm{J}, \mathrm{L}^2)^2=\Delta_2$. Hence shew that the discriminant $(\mathrm{D},\mathrm{D})^2$ of D is $\frac{\Delta}{4}+\frac{\Delta_1}{3}-\frac{4}{27}\Delta_2$.

[1] For the general determination of the irreducible system we may take $f=\lambda x_1^3+\mu x_2^3$, $\mathrm{L}=ax_1+bx_2$. We then have

(2) $\mathrm{H}=2\lambda\mu\, x_1x_2$	(9) $\mathrm{L}'_1=\lambda\mu\,(b^2\lambda x_1-a^2\mu x_2)$.
(3) $\mathrm{J}=\lambda b x_1^2-\mu a x_2^2$	(10) $\Delta=-2\lambda^2\mu^2$
(4) $f'=\lambda\mu\,(\lambda x_1^3-\mu x_2^3)$	(11) $\Delta_1=-2\lambda\mu ab$
(6) $\mathrm{J}'=\lambda\mu\,(\lambda b x_1^2+\mu a x_2^2)$	(12) $\Delta_2=\lambda b^3-\mu a^3$
(7) $\mathrm{L}_1=b^2\lambda x_1+a^2\mu x_2$	(13) $\Delta_3=\lambda\mu\,(\lambda b^3+\mu a^3)$
(8) $\mathrm{L}_2=-\lambda\mu\,(ax_1-bx_2)$	

Ex. (10). By definition Δ_2 is the resultant of f and L and Δ_3 of f' and L' *i.e.*, $\Delta_3=0$ is the condition that the roots of the quartic fL may form a harmonic range. Also Δ_1 is the resultant of H and L *i.e.*, $\Delta_1=0$ is the condition of self-apolarity of the quartic fL. Hence the absolute invariant of the quartic fL may be taken $\dfrac{\Delta_2^2}{\Delta_1^3}$.

Ex. (11). Shew that Δ_2 is also the determinant of the three quadratics L^2, H and J. Hence $\Delta_2=0$ is the condition that there may exist a linear relation between L^2, H and J.

Ex. (12) Establish the syzygy $\Delta_2 L+\Delta_1 L_1-\Delta_2 L_2=0$,

Ex. (13). Also the syzygy $2\Delta_2^2+\Delta_1^3+\Delta\Delta_2^2=0$.

Equivalance of two algebras A, A' under transformation of units.

A *necessary* condition of the transformability of the algebra A into the algebra A' is obviously the equality of the absolute invariants, *viz.*,

$$\Delta:\Delta_1:\Delta_2=\Delta':\Delta'_1:\Delta'_2$$

We shall shew that this condition is certainly sufficient if none of the above invariants vanish.

Denote covariants of A' by dashes. It follows immediately from Ex. (10) and Ex. (13) that the absolute invariants of the quartics fL and f'L' are equal. Hence when $\Delta, \Delta_1, \Delta_2$ do not vanish for either algebra, there exists certainly a transformation M of determinant m of $x_1\ x_2$ which transforms L to L' and f to kf' (say).

Now $k^4\Delta'$=discriminant of $kf'=m^6\Delta$,

$k^2\Delta'_1$=invariant corresponding to Δ_1 of kf' and $L'=m^4\Delta_1$

$k\Delta'_2=$ „ „ Δ_2 „ „ $=m^3\Delta_2$

Hence $$\frac{k^4}{m^6}=\frac{k^2}{m^4}=\frac{k}{m^3}\text{ so that }k=m.$$

Thus M transforms L into L' and f into mf'.

Now if we transform the units of A by the conjugate of the inverse of M, x_1, x_2 will be transformed by M. Since it has been shewn that M transforms L, f into L_1, mf' it follows that the transformation of units transforms the algebra A' determined by L' and f'.

Cor. The proof of the theorem turns upon the fact that $k=m$ which could be deduced from the single equation $\dfrac{k^2}{m^4}=\dfrac{k}{m^3}$.

Hence the theorem remains true even when Δ vanishes for both the algebras provided the other two invariants do not vanish and

$$\Delta_1 : \Delta_2 := \Delta'_1 : \Delta'_2.$$

If one of the invariants Δ, Δ_1, Δ_2 other than Δ vanishes for both algebras, they are not necessarily equivalent. *A fortiori* the same thing is true if more than one of the invariants vanish.

III. *Special numbers of the algebra.*

(a) *Semi-units.*

The semi-units are the numbers which are equal to their own squares. From Ex. (1) and the definition of the covariant f it follows that the semi-units satisfy the scalar equation $f(x)=0$. There are therefore not more than *three* semi-units and in general there are three. The exception arises when $D(x)$ is a perfect square so that the nil-factors coalesce into a single nil-potent number t satisfying the equation $t^2=0$. It is obvious that the number t also satisfies $f(t)=0$ and will therefore account for one or more of the roots of $f(x)$. Thus there will no longer be three semi-units, one or more of them being absorbed by t The number t itself is not a semi-unit but the number pt where p is an infinite scalar, my be considered to be a semi-unit.

The nilpotent number t may be called an *improper semi-unit* thus permitting us to say, that there always exist three semi-units either proper or improper.

Ex. (14) It appears form the above that the covariants f and D are so related, that whenever D has a repeated root, this repeated root is a root of f. Hence *the resultant* R of f *and* D *contains the discriminant of* D *as a factor.* Since R should be of the 8th degree in the constants of the algebra, the other factor should be a linear function of Δ, Δ_1, Δ_2 with numerical co-efficients. To obtain this latter, we consider an algebra for which D is not a square but has a factor in common with f, *e.g.*, the algebra

$$e_1^2=\lambda_1 e_1,\ e_1 e_2=0,\ e_2^2=\nu_1 e_1+\nu_2 e_2.$$

We find easily that $R=(27\Delta+36\Delta_1-16\Delta_2)(27\Delta+9\Delta_1+2\Delta_2)$ the first factor representing the discriminant of D (Ex. (9)).

In the following α, β, γ, are supposed to be proper semi-units.

Ex. (15) $L(\alpha)-D(\alpha)=1$. This follows from the identical equation.

Ex. (16) $D(\beta, \gamma)=\frac{1}{4}$. For $(\beta+\gamma)(\beta-\gamma)=\beta^2-\gamma^2=\beta-\gamma$.

But if $xy=x$, than from the identical equation $y=\dfrac{xL(x)-x^2}{D(x)}$.

Hence using previous exercise and Ex. (2) we find

$$(\beta+\gamma)\ (1-2\mathrm{D}\ (\beta,\gamma))+2\beta\gamma\ (2\mathrm{D}\ (\beta,\ \gamma)-1)=0$$

or $$(\beta-\gamma)^2\ (1-2\mathrm{D}\ (\beta,\gamma)=0,\text{ so that } \mathrm{D}(\beta,\gamma)=\tfrac{1}{2}.$$

Ex. (17) $\beta\gamma=\beta\mathrm{D}_\gamma\ +\gamma\mathrm{D}_\beta$.

Ex. (18) From Ex. (3) we have $\Sigma\mathrm{D}(a)=4\mathrm{D}(a)\ \mathrm{D}(\beta)\ \mathrm{D}(\gamma)+1$.

Ex. (19) Let $a\alpha+b\beta+c\gamma=0$ be the linear relation between the semi-units. Squaring we have $\Sigma a\alpha\ (a+2b\ \mathrm{D}(\beta)+2c\ \mathrm{D}(\gamma))=0$:

so that $$\frac{a}{1-2\ \mathrm{D}(a)}=\frac{b}{1-1\ \mathrm{D}(\beta)}=\frac{c}{1-2\ \mathrm{D}(\gamma)}.$$

Hence we have identically $\Sigma(1-2\ \mathrm{D}(\beta))\ (1-2\ \mathrm{D}(\gamma))\alpha\equiv0$.

Ex. (20) *To investigate the conditions under which the algebra has a principal unit.* If α be the principal unit, then α must be one of the semi-units. Also α can not be a nilfactor so that $\mathrm{D}(\alpha)\neq0$. If β, γ be other semi-units, we have from Ex. (17)

$$\alpha\beta=\beta=\ \alpha\ \mathrm{D}(\beta)+\beta\ \mathrm{D}(\alpha).$$

Hence $\mathrm{D}(\beta)$ and $\mathrm{D}(\gamma)$ are zero, so that β, γ are the nilfactors and $\mathrm{D}(\alpha)=1.$*

From $\mathrm{D}(\alpha)=1$ it follows $\alpha\neq\beta$, $\alpha\neq\gamma$; for if $\alpha=\beta$, $\mathrm{D}(\alpha)$ would be $\frac{1}{2}$ (Ex. 16). We have thus the necessary conditions $\beta\gamma=0$, $\alpha\neq\beta$, $\alpha\neq\gamma$. These conditions are also sufficient for if $\beta\neq\gamma$, Ex. (18) shews that $\mathrm{D}(\alpha)=1$, from which we find $\alpha\beta=\beta$, $\alpha\gamma=\gamma$ shewing that α is a principal unit, The same thing is true by continuity when $\beta=\gamma$.

Ex. (21) In the previous exercise, shew that (1) $\alpha=\beta+\gamma$, (2) $\mathrm{L}(\beta)=\mathrm{L}(\gamma)=1$. Hence $\mathrm{L}(\beta-\gamma)=0$, so that for an algebra with a principal unit $f\mathrm{L}$ is *harmonic* and therefore $\Delta_3=0$. Hence by Ex. (12) $\Delta_2\mathrm{L}_1=\Delta_3\mathrm{L}_2$. Shew that the principal unit whenever it exists satisfies the equations $\mathrm{L}_1(\alpha)=\mathrm{L}_2(\alpha)=0$.

(b) *The Normal number.*

A number e for which $\mathrm{L}(e)=0$ will be called *a normal number*. Obviously all the numbers e are congruent to any one of them.

Ex. (22) If a, b are any two numbers of the algebra, $\mathrm{L}(b)\ a-\mathrm{L}(a)\ b$ is a normal number. The normal number e satisfies the equation

$$e^2=-\mathrm{D}(e)e.$$

Ex. (23) e. $ex=-\mathrm{D}(e)x$. This is seen by taking $x=pe+qe^2$ where p, q are scalars and using last exercise.

* The theorem $\mathrm{D}(\alpha)=1$ for a principal unit may be generalised :—The right-hand and left-hand determinants of the principal unit of any n-ary linear algebra are each equal to unity—neither the associative nor the commutative property being assumed (Vide Dickson).

Hence the solution of $ex=a$ is $x=\dfrac{-ea}{D(a)}$.

(c) *Higher Semi-units.*

Generally a number E of the algebra equal to E^{n+1} may be termed a semi-unit of order n. Thus what we have already termed semi-units would be semi-units of order 1, while the normal number would be a number congruent to the semi-units of order 2. Writing $E=E_1e_1+E_2e_2$ and $E^{n+1}=f_1(E_1, E_2)e_1+f_2(E_1, E_2)e_2$ where $f_1 f_2$ are of order $n+1$, it is apparent that these are $n+2$ incongruent semi-units of order n. Now a semi-unit of order n is also a semi-unit of orders $2n$, $3n$,... Hence if we term '*special* semi-units of order n,' those semi-units which are not at the same time semi-units of lower order, it would obviously follow that the number of special incongruent semi-units of order $n(>1)$ is equal to the number of numbers $<n$ and prime to it.

It is also easy to prove that if E be a semi-unit of order n, then $E_n(x)=x$ where $E_1(x)=Ex$, $E_k(x)=E.\ E_{k-1}(x)$. It will be noticed that the equations that give the special semi-units may be obtained from the identical equation and therefore depend only on L and D.

IV. Special Algebras for which one of the covariants f, D, L, vanishes identically.

(1) *Singular Algebras.*

These are the algebras for which all the three covariants vanish. This happens either when $f\equiv0$, $D\equiv0$, or when $f\equiv0$, $L\equiv0$. All the constants of a singular algebra vanish. Hence the product of any two numbers is zero, so that multiplication in a sigular algebra is not a significant operation. These algebras may accordingly be dismissed with the remark that they are of rank 2 ($\because x^2=0$).

(2) *Nilpotent algebras of rank 3.*

For these algebras $f\neq0$, $D=0$, $L=0$. The identical relation becomes $x^3=0$, so that every member is nilpotent of rank 3. Since $L\equiv0$, $D=\frac{1}{4}H$ (Ex. 8), hence $H\equiv0$ so that f is a perfect cube, say p_x^3. D, though identically zero, should, since it$=\frac{1}{4}H$, be considered as factorisable into p_x and an arbitary vanishing linear form. Hence if e is the (improper) semi-unit $e\times$any number $=0$. Hence the algebra may be taken in the form $e_1^2=e_1e_2=0$, $e_2^2=e_1$, $(e_1=e)$.

All the invariantss of a nilpotent algebra vanish. It is also seen that a nilpotent commutative algebra is necessarily associative.

(3) *Unitary Algebras.*

These are the algebras for which $f\equiv0$, $L\neq0$, $D\neq0$. For a unitary algebra $D=\frac{2}{3}L^2$ (Ex. 8). Hence the square of the normal number is zero.

Since f is zero identically there are an infinite number of semi-units. Now if x is a semi-unit of the algebra $D(x)=D(x, x)=\frac{1}{3}$, (Ex. 16); hence $L(x)=\frac{2}{3}$, (Ex. 16). Hence any number e such that $L(e)=\frac{2}{3}$ is a semi-unit, *i.e.*, if x be any number whatsoever, $\dfrac{3x}{2L(x)}$ is a semi-unit. Thus the identical equation fo the unitary algebra is $x^2=\dfrac{2L(x)}{3}\cdot x$. A unitary algebra is therefore of rank 2.

Ex. (24) All the invariants of a unitary algebra vanish. If x, y be any two number of the algebra $3xy=xL(y)+y\ L(x)$ and if e be the normal number $3ex=eL(x)$.

Ex. (25) A unitary algebra may be reduced to the form

$$e_1{}^2=0;\ e_1e_2=e_1,\ e_2{}^2=2e.$$

In connection with the form of the identical equation of a unitary algebra, it will be noticed that $x^2-\dfrac{2L(x)}{3}\ x$ is a factor of $x^3-x^2\ L(x)+\frac{2}{9}(L(x))^2\ x$, where x is regarded as a scalar. This verifies a general theorem (Vide Dickson) in Linear Algebras to the effect that the rank or identical equation is obtained by equating to zero one of the factors of the characteristic function.

(4) *Null Algebras.* $D\equiv0$, $f\neq0$, $L\neq0$.

Since D=linear function of L^2, H and $J\equiv0$, $\Delta_3=0$ (Ex. 11), we notice that f can not be a perfect cube; for then H would be zero and J therefore a multiple of L^3. Hence f would be divisible by L^3 and therefore since it is a cube, by L^3. J and therefore L would then vanish.

Since $\Delta_3=0$, fL is harmonic. Hence if f has a repeated factor the repeated factor must be L.

From $D(x)=0$, Ex. (5) gives $(x^2)^2=x^2\ L(x^2)$.

In other words, the square of any number x (and $\therefore$ also the product of any two numbers) is congruent to a semi-unit—say e—which by continuity is the same for all numbers x. If $\Delta\neq0$, the squares of each of the other semi-units which are distinct from e and from each other, should be congruent with e and also with themselves. Hence

these squares must vanish shewing that the other semi-units are improper. It is obvious further that if e' be the normal number $ee'=0$. If $\Delta=0$, either the improper semi-units $e_1 e_2$ may coalesce or one of them may coalesce with e. In the former case e remains a proper semi-unit and the normal number is the repeated improper semi-unit. In the latter case e is an improper semi-unit and also the normal number.

This investigation shews that in a Null Algebra $D(x)$ though identically vanishing should be regarded as remaining apolar to a fixed quadratic factor of f, the number e being the semi-unit corresponding to the remaining factor.

Ex. (26) For a null algebra with vanishing Δ all the invariants vanish. If $\Delta\neq0$, $\Delta:\Delta_1:\Delta_2=-8:18:27$. The second part may be obtained from any two of the equations $(D, D)^2=0$, $(D, L^2)^2=Q$, $(D, H)^2=0$, $(D, J)^2=0$.

Ex. (27) A null algebra can be reduced to one of the following forms:

$$(1)\quad e_1^2=e_2^2=0,\ e_1e_2=e_1+e_2\ (\Delta\neq0)$$

$$(2)\quad e_1^2=e_1e_2=0,\ e_2^2=e_2.\ (\Delta=0)$$

$$(3)\quad e_1^2=e_2^2=0,\ e_1e_2=e_1\ (\Delta=0).$$

(5) *Normal Algebras.* $L\equiv0$, $f\neq0$, $D\neq0$.

Here every number is a normal number. Since $L\equiv0$, $D=\frac{1}{2}H$. From Ex. (6) we find that each nil-factor is congruent to the square of the other. If $\Delta\neq0$ there exist three proper semi-units. If $\Delta=0$, H and $\therefore$ D is a factor of f and there is only one proper semi-unit. f can not be a perfect cube for then $D=0$.

If x, y, z, are proper semi-units, $D(x)=-1$; Hence $x+y+z=0$, $x=yz$, $y=zx$, $z=xy$.

Since every number is normal $a\cdot ab=-D(a)\,b$ for any two numbers a, b. Hence the solution of $ax=e$ is $x=\dfrac{-ae}{D(a)}$. If e is a nilfactor, x must be congruent to the other nilfactor, for $e\cdot ea=-D(e)\,a=0$. Hence ax can be a nilfactor only when either a or x is a nilfactor; in other words $D(ax)=0$ when and only when either $D(a)$ or $D(x)=0$. Since $D(a)$ is absolute covariant quadratic in (a_1, a_2) and (x_1, x_2) we must have identically $D(ax)=KD(a)\,D(x)$ where K is numerical. By taking a, x to be semi-units we reach the theorem.

If a, x be any two number of a normal algebra $D(ax)=-D(a)D(x)$.

Ex. (28) By differentiation deduce

$$D(ax, by)+D(ay, bx)=-2D(a, b)\,D(x, y).$$

Putting $a=x$, $b=y$, we get $D(x^2, y^2)-D(x)D(y)=-[D(x, y)]^2$.

Ex. (29) By differentiating the identical equation, shew

$$(1) \quad xy^2=D(y)x-2y\, D(x, y).$$

$$(2) \quad x.\,ay-y.\,ax=2[D(a, y)x-D(a, x)y].$$

The Quasi-Associative Property.

Squaring both sides of the identity $x^2D(y)+y^2\,D(x)=2\,D(x, y)\,xy$, we have $4[D(x, y)]^2\,(xy)^2=(x^2)^2\,D_y^{\,2}+(y^2)^2D_x^{\,2}+2x^2y^2D_xD_y$.

$$=-(x^2)^2\,D(y^2)-(y^2)^2\,D(x^2)+2x^2y^2\,D(x)\,D(y).$$

$$=2\,x^2y^2\,(D_x\,D_y-D(x^2, y^2))$$

$$=4[D(x, y)]^2\,x^2y^2 \text{ (Ex. 26)}.$$

Hence $(x\,y)^2=x^2y^2$. ... ... (1)

Now multiplying the identities

$$2D(x, y)\,xy=x^2\,D(y)+y^2\,D(x)$$

$$2D(a, b)\,ab=a^2D(b)+b^2\,D(a).$$

We have $4abxy\;D(a, b)\,D(x, y)=a^2x^2D_yD_b+a^2y^2\,D_xD_b+b^2x^2\,D_aD_y$
$+b^2y^2D_aD_x$.

Hence unsing Ex. (26) and result (1) above, we have

$$-2abxy\,(D(ax, by)+D(ay, bx)+D(ab, xy)=\Sigma a^2x^2D_yD_b.$$

Thus ab, xy is a symmetric function of a, b, x, y ; so that

$$ab\cdot xy=ax\cdot by=ay\cdot bx$$

Ex. (30). Prove that the quasi-associative property may be stated thus : Let $abcx$ be a successive product, multiplication beginning from the right, the multiplier at each stage being a single symbol. Then $abcx=cbax$.

More generally the value of the successive product $a_na_{n-1}\,..a_2a_1x$ is not altered if we permute anong themselves the numbers $(xa_1a_3\,..)$ or the numbers $(a_2a_4a_6\;\;)$.

Ex. (31). Express the product of two successive products as a successive product.*

Ex. (32). Prove from Ex. (28) that $abcx=\lambda x$ where λ is a number of the algebra not depending on x but only on a, b, c. Extend the theorem to the case when there is any odd number of successive multipliers.

* For theorems on successive products and other developments see the author's paper 'Properties of Equations in the Algebra of the Nodal cubic' Journal of the Indian Mathematical Society. Vol. X, No. 4.

Ex. (33). All the invariants of a normal algebra vanish. Prove that a normal algebra can be reduced to one of the forms

$$(1) \quad e_1^2=e_2,\ e_2^2=e_1,\ e_1e_2=0,$$

$$(2) \quad e_1^2=e_1,\ e_2^2=0,\ e_1e_2=-e_2.$$

V. Algebras of rank 2.

By the general theorem, referred to already, the possible identical equations defining an algebra of rank 2, must arise from the factors of the expression $x^3-x^2L(x)+x\,D(x)$, where x is regarded momentarily as a scalar. The possible rational factors of the second degree of this expression are obtained: (1). By removing the factor x. But the only case in which the remaining factor can lead to an identical equation is when $x\equiv\epsilon x$ where ϵ is a definite number of the algebra. We are thus led to the algebra with a principal unit—which is obviously of rank 2. (2) When there is no number ϵ, the above factorisation does not lead to an identical equation and we must then combine x with one of the factors of $x^2-x\ L(x)+D(x)$, these factors being supposed rational in x_1, x_2 and the constants of the algebra. The condition for the existence of such factors is that L^2-4D should be the square of a rational function which has obviously to be an absolute linear covariant; this shews that D should be a multiple of L^2 and the identical equation should therefore be of the form $x^2=KL(x)\ x$, (where K is a scalar) possibly degenerating into $x^2=0$. For the first form, every number is a semi-unit, so that $f\equiv0$, leading to a unitary algebra which was already shewn to be of rank 2 with the identical equation $x^2=\frac{2}{3}L(x)\ x$; the second form leads to the singular algebra. Hence *the only algebras of rank 2 are the algebras with a principal unit, the unitary algebras and the singular algebra.*

Ex. (34). $\Delta_3=0$ for an algebra of rank 2.

VI. Algebras with a principal unit.

It was shewn that $\Delta_3=0$ for algebras with a principal unit, and that the principal unit satisfies the equation $L_2(a)=L_1(a)=0$. We proceed to determine the absolute invariants of these algebras.

When $\Delta_3=0$, L_2 is a factor of f, for $(f, L_2^3)^3=-\frac{1}{2}\Delta\Delta_3$; and D is apolar to LL_2, for L^2 and H are always apolar to LL_2 and when $\Delta_3=0$, D can be expressed as a linear combination of L^2 and H. Hence the conditions already obtained for an algebra with a principal

unit are analytically equivalent to the following :—

(1) $\Delta_3=0$,

(2) $R=0$ where R is the resultant of f and D,

(3) $(D, L_2{}^2)^2\equiv\frac{2}{9}\Delta_1{}^2+\frac{1}{4}\Delta\Delta_1+\frac{1}{6}\Delta\Delta_2\neq 0$.

From (1) and the syzygy of Ex. (13), we have $\Delta_1{}^3+\Delta\Delta_2{}^2=0$. Solving this equation in combination with

$$R\equiv(27\Delta+36\Delta_1+16\Delta_2)\,(27\Delta+9\Delta_1+2\Delta_2)=0,$$

we obtain three sets of values .

$$\Delta:\Delta_1:\Delta_2=-8:18:27,$$

$$\frac{-64}{9}:4:-3,$$

$$1:-9:27.$$

of these the first two values do not and the last does satisfy condition (3). Hence for an algebra with a principal unit $\Delta:\Delta_1:\Delta_2=1:-9:27$.

Ex. (35). For an algebra with a principal unit prove the identity

$$DL_2\equiv\left(\frac{\Delta_2}{9}-\frac{\Delta_1}{2}\right)f.$$

Ex. (36) Shew that there are two non-equivalent types of algebras with a principal unit according as $\Delta=$ or $\neq 0$. In the latter case the algebra can be reduced to $e_1{}^2=e_1$, $e_2{}^2=e_2$, $e_1e_2=0$ and in the former, to $e_1{}^2=0$, $e_1e_2=e_1$, $e_2{}^2=e_2$.

VII. *Associative Algebras.*

It is easy to see that $(x^2)^2=x^4$ is necessary as well as sufficient condition that a binary commutative algebra be in addition associative. Using Ex. (5) and $x^4=x^3\ L(x)-x^2\ D(x)=x^2\ ((L(x)^2-D(x))-x\ D(x)\ L(x)$, this reduces to $L(x^2)+D(x)=(L(x))^2-D(x)$,

$$2D(x^2, x)=D(x)\ L(x).$$

Hence using Ex. (8), we find $2D=\frac{1}{3}L^2-J$ or $\frac{1}{9}L^2+H+\frac{1}{3}J=0$. Hence $\Delta_3=0$ and $\Delta_1+\frac{1}{3}\Delta_2=0$, so that we obtain $\Delta:\Delta_1:\Delta_2=1:$

$-9:27$. *Hence unless all the invariants vanish, the algebra has a principal unit.* Now when the invariants vanish in the ratio $1:-9:27$ and $L \neq 0$, the resulting algebra has still a principal unit being in fact the first of the two types considered in Ex. (34). If however $L = 0$, it immediately results that D and H are each zero and therefore f is either a vanishig or non-vanishing perfect cube. The algebra loses its character of possessing a principal unit and becomes nilpotent of rank 2 or 3. Thus *the only binary commutative aossociative algebras are the two types of algebras with a principal unit and the nilpotent algebras of rank 2 or 3.*

We might have also obtained the result by using the theorem that an associative algebra has a principal unit unless the characteristic determinant is identically zero (Dickson, Linear Algebras).

Ex. (37) Investigate the number of non-equivalent types of algebras for which all the invariants vanish.

VIII. Primitive Algebras.

These are the algebras which possess an infinity of pairs of numbers each of which is the square of the other.

For a primitive algebra $(x^2)^2$ is identically congruent to x, so that $L(x^2)^2 + D(x) \equiv 0$ or $16L^2 + 9H + 12J = 0$. Hence $\Delta_3 = 0$ and $\Delta : \Delta_1 :$

$\Delta_2 = -\frac{64}{9} : 4 : -3$. Substituting these values in the discriminant of D,

we find that D is a perfect square. It is obvious that D cannot be identically zero.

Otherwise—the pairs of numbers each of which is the square of the other must be connected by an involuteric relation the self-corresponding numbers of which must be two distinct proper semi-units e_1, e_2 say. If the third semi-unit be $e_3 = \lambda e_1 + \mu e_2$, then since $\Delta_3 = 0$, $\lambda e_1 - \mu e_2$ must be the normal number. From $L(x^2) + D(x) = 0$, it follows that the square root of the normal number must be a nilfactor which must be a repeated nilfactor—since obviously any number of a primitive algebra has only two square roots which are mutually congruent. Thus D is a perfect square and the corresponding nilpotent number is the third semi-unit e_3. The algebra may thus be reduced to the from

$$e_1^2 = e_1,\ e_1 e_2 = -\tfrac{1}{2}(e_1 + e_2),\ e_2^2 = e_2.$$

Ex. (38). Investigate the most genaral algebra in which any number has only two square roots which of course are mutully congruent.

Ex. (39) Investigate the general algebra for which the squares of any two numbers are congruent.

Ex. (40). Prove that a primitive algebra can be reduced to the from $e_1^2=e_2$, $e_2^2=e_1$, $e_1e_2=\mu_1e_1+\mu_2e_2$, $(\mu_1\mu_2=\frac{1}{4})$

Ex. (41). For a primitive algebra though D is a perfect square yet $\sqrt{D}$ is not a rational covariant. Shew in fact

$$4L_2^2+3\Delta D\equiv 0.$$

IX. Algebras for which $\Delta_3=0$.

It will not have escaped notice, that Δ_3 was zero, for all the special algebras we have so far considered. Hence the algebras for which $\Delta_3=0$ would form an exceptionally important sub-class of commutative algebras in general. In what follows we exclude the cases in which any of the covariants f, D, L vanishes identically.

(1). fL is a perfect fourth power. In this case all the invariants vanish and the algebra reduces to $e_1^2=0$, $e_1e_2=e_1$, $e_2^2=e_1+2e_2$.

(2). f is a perfect cube and L not a factor of L. Here all the invariants except Δ_3 vanish. The algebra may be reduced to

$$e_1^2=e_2^2=e_1,\ 2e_1e_2=e_2.$$

This algebra will be called the inflexion algebra (see next para).

(3). f has a squared factor which since $\Delta_3=0$ must be L^2. All the invariants vanish. D is a multiple of L^2 and the algebra reduces to

$$e_1^2=e_1,\ e_2^2=0,\ e_1e_2=k\,e_2.$$

when $k=1$, this algebra has a principal unit. The absolute invariants of the algebra are finite, being rational functions of k

(4). f has no repeated root. Of the three semi-units, one is *special* in that it is associated with L in the harmonic range formed by the roots of fL ; and this special semi-unit satisfies the equation $L_2(x)=0$. Also from $\Delta_3=0$ it follows that D is apolar to LL_2 ; hence both the nonspecial semi-units are *proper*. The algebra may be reduced to

$$e_1^2=e_1,\ e_2^2=e_2,\ e_1\,e_2=\mu(e_1+e_2).$$

The absolute invariants are rational functions of μ. Thus

$$\Delta:\Delta_1:\Delta_2=\frac{2}{27}(2\mu-1)^4:-\tfrac{2}{3}(\mu+1)^2(2\mu-1)^2:-2(2\mu-1)(\mu+1)^3.$$

Ex. (42). Investigate the algebra which contains a number which remains congruent to itself after being multiplied by any other number of the algebra.

Ex. (43). Investigate the algebra which contains a number a such that ax is always congruent to x.

X. *Geometrical Representation.*

The intrinsic or invariant properties of a binary commutative algebra are completely determined from the ratios $\Delta:\Delta_1:\Delta_2$ provided $\Delta_1, \Delta_2 \neq 0$. Thus each algebra or rather algebra-type for which $\Delta_1, \Delta_2 \neq 0$, may be put in reversible one—to—one correspondance with the point in two-dimensional space whose homogenous coordinates are $\Delta, \Delta_1, \Delta_2$. If Δ_1 or Δ_2 vanishes, it is still true that only one point corresponds to the algebra-type, though the same point corresponds to more than one type. The algebras for which all the invariants vanish are excluded from this correspondence. The algebras for which $\Delta_3=0$, would be represented by points on the cusped cubic $\Delta_1^3+\Delta\Delta_2^2=0$. The cusp Δ represents the normal algebra, the point of inflexion Δ_2 represents the inflexion-algebra, *viz.*, the algebra (2) IX.

Consider now the factors of the resultant of D f. The factor $27\Delta+36\Delta_1-16\Delta_2$ which is the discriminent of D corresponds to a straight line PQ and the other factor $27\Delta+9\Delta_1+2\Delta_2$ to another straight line PR. P is the point $(-8, 18, 27)$ and therefore lies on the cusped cubic and is in fact the representative point of the null algebra. It is further verified that PQ is the tangent to the cubic at P and PR is the tangent at R. It is also verified that Q corresponds to the primitive algebra and R to the algebra with a principal unit.

Ex. (44). Examine how far an algebra-type is determined by a given identical equation.

Giving the identical equation is tantamount to giving D and L. Hence writing

$$\lambda=(D, D)^2,\ \mu=(D, L^2)^2,$$

we have
$$\lambda=\frac{1}{4}\Delta+\frac{1}{3}\Delta_1-\frac{4}{27}\Delta_2, \quad \dots (1)$$

$$\mu=\frac{1}{2}\Delta_1-\frac{1}{3}\Delta_2, \quad \dots (2)$$

Hence $$\mu\left(\frac{\Delta}{4}+\frac{\Delta_1}{3}-\frac{4\Delta_2}{27}\right)-\lambda\left(\frac{\Delta_1}{2}-\frac{\Delta_2}{3}\right)=0.$$

Thus the representative point of the algebra-type traces a straight line passing through the intersection of (1) and (2), *i.e.*, passing through P the representative point of null algebras.

NOTES AND NEWS

The following lines from the inaugural lecture delivered by Prof. G. H. Hardy before the University of Oxford on some famous problems of the theory of numbers and in particular Waring's problem will be read with interest :—" If I am asked to explain how, and why the solution of the problems which occupy the best energies of my life is of importance in the general life of the community, I must decline the unequal contest....I suppose that every mathematician is sometimes depressed, as certainly I often am myself, by a feeling of helplessness and futility. I do not profess to have any very satisfactory consolation to offer. It is possible that the life of a mathematician is one which no perfectly reasonable man would elect to live. There are, however one or two reflections from which I have sometimes found it possible to extract a certain amount of comfort. In the first place, the study of mathematics is, if an unprofitable, a perfectly harmless and innocent occupation, and we have learnt that it is something to be able to say that at any rate we do no harm. Secondly the scale of the universe is large, and, if we are wasting our time, the waste of the lives of a few University dons is no such overwhelming catastrophe. Thirdly, what we do may be small, but it has a certain character of permanence ; and to have produced any thing of the slightest permanent interest, whether it be a copy of verses or a geometrical theorem, is to have done something utterly beyond the powers of the vast majority of men. And, finally, the history of our subject does seem to show conclusively that it is no such mean study after all. The mathematicians of the past have not been neglected or despised ; they have been rewarded in a manner indiscriminating perhaps, but certainly not ungenerous." Indians born of a race of philosophers will perhaps add another clause to these arguments and it is this that the solution of an absorbing mathematical problem is perhaps the easiest process of attaining that mental *Samadhi* which has been declared to be the aim of life and that it is a joy by itself.

The issue of the *Proceedings of the London Mathematical Society*, Vol. 19, Part 6, contains an obituary notice of the Indian

mathematician Mr. S. Ramanujan, F.R.S., by Prof., G. H. Hardy and perhaps no one else is in a better position to tell the story of Ramanujan's life than Prof. Hardy. "I have often been asked," writes Prof. Hardy, "whether Ramanujan had any special secret, whether his method differed in kind from those of other mathematicians, whether there was anything really abnormal in his mode of thought. I cannot answer these questions with any confidence or conviction; but I do not believe it. My belief is that all mathematicians think, at bottom in the same kind of way, and that Ramanujan was no exception. He had, of course, an extraordinary memory. He could remember the idiosyncrasies of numbers in an almost uncanny way. It was Mr. Littlewood who remarked that every positive integer was one of his personal friends...It was his insight into algebraical formulae, transformations of infinite series, and so forth, that was most amazing. On this side most certainly I have never met his equal, and I can compare him only with Euler and Jacobi. He worked, far more than the majority of modern mathematicians, by induction from numerical examples; all of his congruence properties of partitions were discovered in this way. But with his memory, his patience, and his power of calculation, he combined a power of generalisation, a feeling for from, and a capacity for rapid modification of his hypotheses, that was often really startling, and made him, in his own peculiar field, without a rival in his day."

The issue of *Mathematische Zeitschrift*, Vol. 9, nos. 1-2 contains an article on "Congruence properties of partitions" by Ramanujan. This is extracted by G. H. Hardy from one of the author's manuscripts which "contains a large number of further results. It is very incomplete,. and will require very careful editing before it can be published in full. I have taken from it," Hardy writes, "the three simplest and most striking results as a short but characteristic example of the work of a man who was beyond question one of the most remarkable mathematicians of his time."

The following advanced courses in Mathematics are offered at the Calcutta University during the academic year 1921-1922:

Pure Mathematics: Mr. S. C. Dhar: Higher Algebra and Plane Trigonometry; Mr. S.M. Gangooly: Higher Plane Curves; Dr. H. Bagchi: Differential Geometry; Mr. M. Gupta: Integral Calculus; Mr. N. K. Majumdar: Differential Equations; Mr. S. Ghosh: Differential Calculus; Mr. M. Ghosh: Spherical-Trigonometry; Mr.

I. B. Brahmachary : Theory of Equations and Algebra of Quantics ; Mr. H. P. Banerjee : Solid Geometry and Calculus of Variations ; Dr. S. D. Mookerjee, and Mr. Gupta : Functions of Real Variables ; Mr. S. Gangooly and Mr. H. P. Banerjee : Functions of Complex variable ; Mr. I. B. Brahmachari and Mr. S. C. Bose : Projective Geometry ; Dr. S. D. Mookerjee and Mr. S. Gangooly : Non-Euclidean Geometry ; Dr. S. D. Mookerjee and Mr. N. K. Majumdar: Theory of Groups ; Mr. N. K. Majumdar and Mr. S. C. Dhar : Finite Differences ; Dr. H. Bagchi and Mr. S. C. Ghosh : Quaternions ; Mr. H. P. Banerjee and Mr. M. Gupta : Theory of Numbers.

Applied Mathematics : Mr. S. P. Das : Statics ; Mr. K. M. Khastgir : Dynamics of a Particle ; Mr. N. R. Sen : Rigid Dynamics ; Mr. B. C. Das : Spherical Astronomy ; Mr. N. N. Sen : Hydrostatics and Hydrodynamics : Dr. S. K. Banerjee : Attraction and Potential ; Dr. D. N. Mallik : Higher Parts of Spherical Astronomy ; Mr. B. B. Datta : Lunar and Planetary Theories ; Dr. S. K. Banerjee : Theory of Elasticity ; Mr. S. C. Kar : Advanced Dynamicces ; Mr. B. B. Datta : Figure of the Earth : Mr. N. R. Sen : Theory of the Tides.

GRAPHIC SOLUTION OF SPHERICAL TRIANGLES WITH APPLICATIONS TO ASTRONOMY.

BY

G. H. BRYAN, Sc.D.,

President of the Institute Aeronautical Engineers.

[Read August 14th, 1921.]

The teaching of astronomy to a class in applied mathematics loses much of its interest and stimulating influence from the fact that the great majority of problems in practical astronomy involve the solution of spherical triangles and that the only methods of solution commonly accessible to students are based on the use of the formulae of spherical trigonometry. In this way the attention of the pupil is diverted to the dredgery of algebraic work, and the geometrical significance of the conclusions is liable to be lost.

It appears to me desirable therefore to substitute geometrical constructions for the trigonometric solutions of the spherical triangles concerned. In this way the pupil will be required actually to draw and measure the angles involved in any problem and he will thus have an opportunity of visualising them instead of merely writing down their numerical values.

The solution of a spherical triangle by constructive geometry involves nothing more than the construction of a few plane triangles and once the geometry of the figures is understood the task should be easy to any student accustomed to work with a ruler, compasses and protractor.

It will be desirable in the first instance to choose problems in which the *sides* of the spherical triangle are less than a right angle. In the later stages of the course, when a triangle occurs, which cannot be solved directly by the present geometrical constructions owing to few of its sides being obtuse, the solution may be made to depend on colunar triangle. For instance in working a problem involving a star whose declination is south we might take the south polar distance instead of the north polar distance as one of the sides of our spherical triangle. The triangle would then be drawn below the horizon but at an advanced stage of the work no very serious difficulty would be experienced.

Solution of Spherical Triangles by Constructive Geometry.

First method. Problems involving one *angle* only of the triangle.

Case I. Given three sides a, b, c, to find one of the angles A.

Construction :—See Fig. 1. Take any convenient length OA_1 and construct the triangles A_1OB_1, A_1OC_1 right angled at A_1 and call these triangles γ and β having $\angle A_1OB_1 = \angle A_1OC_1 = b$. Construct the triangle a or $C_1\ OB_2$ having $OB_2 = OB_1$ and $\angle C_1OB_2 = a$. Construct the triangle δ or $A_1B_3C_1$ having $A_1B_3 = A_1B_1$ and $C_1A_3 = C_1B_2$. Then shall $C_1A_1B_3$ be the required angle of the spherical triangle.

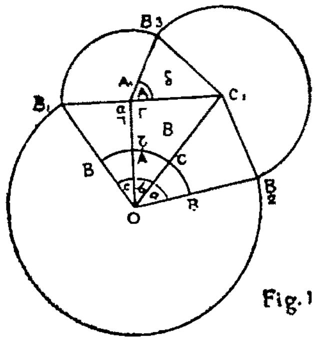

Fig. 1

Experimental Verification. With O as centre and any radius r describe a circular arc and let its intersections with OB_1, OA_1, OC_1 and OB_3 be called B, A, C and D. Cut the group of four triangles out and fold them about the sides of the inner triangle OA_1C_1 so as to form a tetrahedron, the vertices B_1, B_2, B_3 being brought into coincidence. The triangle A B C formed by the circular arcs will be the required spherical triangle and on a sphere of radius r about O, $C_1\ A_1\ B_3$ will become the dihedral angle between the corresponding faces of the tetrahedron that is the required angle A of the spherical triangle.

After the figure has been folded up, unfold it and lay it out and fold it again several times until you clearly understand the construction.

Limitations. Since the triangles A_1OB_1 and A_1OC_1 are right-angled the construction does not hold good without introducing some modifications unless the sides b, c containing the required angle A are each less than 90°. But a may be either acute or obtuse.

Case II. Given two sides b, c and the included angle A, to find the third side a.

Construction. Construct the triangles γ and β as before, but next construct the triangle δ having $A_1\ B_3 = A_1\ B_1$ and $\angle C_1A_1B_3 = A$.

Then construct the triangle a having $OB_2 = OB_1$ and $C_1B_1 = C_1B_3$. The angle C_1OB_2 will be the required third side a.

The experimental verification and actual construction of the triangle on a sphere of any required radius will be the same as before.

Method 1a. Right angled triangles in which one of the angles only is given or is required.

In this case (Fig. 2) if C is the right angle of the spherical triangle, the angles $A_1C_1B_1$ and OC_1B_2 will be right angles, and it follows that each of the four triangles in figure 1 is a right angled triangle. If therefore any two parts of the spherical triangle are given we can construct two of the four triangles. There are however two cases to consider.

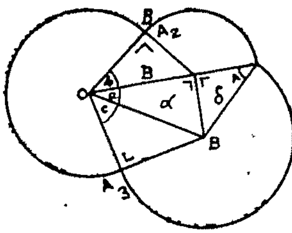

Fig. 2

Case Ia. Whed b is one of the two given parts.

In this case the middle triangle β will be one of the known triangles, and the other one will be adjacent to it. There is no difficulty here about completing the construction of the remaining right angled triangles.

Case IIa. When the given parts are c and A.

In this case the right angled triangle γ is first constructed having $A_1OB_1 = C$, then the triangle $A_1C_1B_3$ is constructed having A_1C_1 in the same straight line with B_1A_1 and with

$$A_1B_3 = A_1B_1, \ \angle C_1A_1B_3 = A \text{ and } \angle A_1C_1B_3 = 90^\circ.$$

Case IIIa. When the given parts are a, c or a, A.

In this case denoting as before the triangles which are to form the faces of the tetrahedron opposite to $A_1B_1C_1O$ by $a_1\beta_1\gamma_1\delta$ we see that in Fig. 1 the known triangles are not adjacent and it is therefore desirable to perform the constructions as shown in Fig. 2, the triangle a now becoming the middle triangle of the figure, and $\llcorner$ denoting right angles.

If a, c are given the order of construction is α, γ, δ, β, while if a, A are given the order is α, δ, β, γ.

Second method.—When *two* of the angles of the spherical triangle are given or are required.

To understand the method it will be best in the first instance to make a copy of Fig. 3 leaving out the circular arcs (with the exception of the arc BACB about O) and to cut out the portion consisting of the triangles α, β, δ, ϵ and the quadrilateral γ, the right angles in the figure being indicated by $\llcorner$ as before.

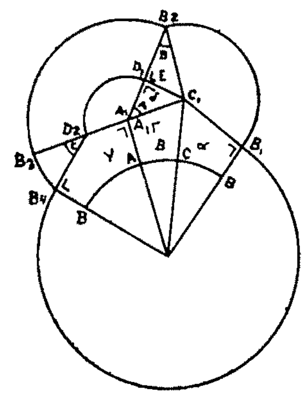

Fig 3.

Taking the quadrilateral γ as base the remaining portions of the figure are to be folded over about the lines OA_1, OC_1, A_1C_1 and C_1D_1, thus bringing D_1 into coincidence with D_2 and B_1, B_3 into coincidence with B_4. The figure will then become a five faced solid, three faces α, β, γ being the faces forming the sides of the spherical triangle, and C_1 D_1 will now be the perpendecular let fall from the point C_1 on the edge OC on the opposite face of the trihedral angle. The sides a, b, c and angles A, B of the spherical triangle are indicated in the figure and the cicular arcs as in the previous case show the lengths which are equal as well as the manner in which they are to be brought into coincidence in constructing the model. The various cases may be dealt with briefly.

Case III. Given two sides a, b, and the angle A opposite one of them.

Consider first the non ambiguous case or the solution of the ambiguous case for which the angle opposite the other given side is acute.

In Fig. 3 first construct the right angled triangles α, β with common hypotenuse OC. Next construct the triangle $A_1C_1B_2$ having angle $C_1A_1B_2 = A$ and $C_1B_2 = C_1B_1$.

Then $A_1B_2C_1$ is the required angle B of the spherical triangle opposite the other given side.

To find the third side of the triangle produce C_1A_1 and first cut off $A_1D_2 = A_1D_1$ and $A_1B_3 = A_1B_2$. Then with centres O, D_2 and radii OB_1 and D_2B_3 describe arcs cutting in B_4. Then A_1OB_4 will be the required third side of the spherical triangle.

The obtuse solution of the ambiguous case can be found *either* by taking the obtuse solution for the triangle $A_1C_1D_1$ *or* by taking the second point of intersection of the circles constructed about O and D_2 in the last step of the construotian. Both methods will evidently lead to the same result. The solution is shown in Fig. 4 which gives both solution. The latters B'', B''', B'' denote the points of construction obtained by taking the obtuse solution of the triangle A_1C_1B'', but it will be seen that both solutions lead to the same circle about D_2 and the same angle A_1OB^{iv} for the third side in the obtuse solution. In this case the model (which is a little more difficult to construct) must be made by taking the acute angled triangle $A_1C_1B_2$ and not A_1C_1B'' and following it so as to double back D_1B_2 into the position D_2B''.

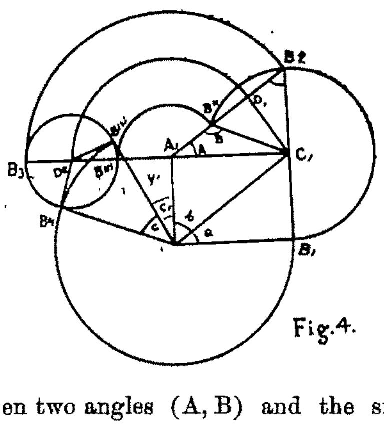

Fig. 4.

Case IV. Given two angles (A, B) and the side opposite one of them, *viz.* *b*.

In this case the triangles A_1C_1D and $A_1C_1B_2$ are constructed from the required points. Then the trinagle *a* is constructed by drawing a circle on OC_1 as diameter to meet in B_1 the circle with centre C_1 and radius CB_2 the rest of the construction is the same as before.

Case V. Given two angles and the side intercepted between them *viz.* A, B and *c*.

First construct the triangle A_1CB_2 having the given angles A, B, and draw the perpendicular C_1D_1 Produce C_1A_1 to B_3 and cut off $AB_3=AB_2$ and $AD_2=AD_1$. Make the angle $B_3D_2B_4$ equal to the given angle c, and let the perpendiculars at B_4 and A_1 to D_2 B_4 and D_2B_1 intersect in O. Finally draw circles centrer O_1, C_1 and radii OB_4 and C_1B_2, and let these circles cut in B_1. Then C_1OA_1 and C_1OB_1 will be the remaining sides of the spherical triangle.

Limitations. In this method the side a, b must be acute but c may be acute or obtuse.

Alternative Solutions of Cases I and II. We have seen that in the first method the sides adjacent to the angle which is given or found must be both less than 90° while in the present method the side c which is adjacent to both angles may be obtuse. This circumstance enables us to employ the second method in cases where the first method fails. In this case the constructions will be as follows.

Case VI. Given two sides and the included angle, *viz.* b, c and A.

Construct the right angled triangle β of fig. 3. Make $C_1A_1D_1$ equal to A and drop C_1D_1 perpendicular on A_1D_1. Produce C_1A_1 to D_2 and make $AD_2=AD_1$. Construct the angle A_1OB_4 equal to the side c, and drop D_2B_4 perpendicular on OB_1. The remainder of the construction now gives no difficulty.

Case VII. Given the three sides a, b, c.

Construct the right angled triangles α, β (fig. 3) and the make angle A_1OB_4 equal to c. Cut off $OB_4=OB_1$ and at B_1 erect a line at right angles to OB_4 meeting OA_1 produced in D_2. The construction may now be completed.

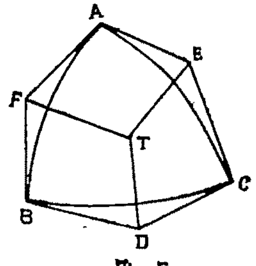

Fig. 5

Case VI. When the three angles are given. (Probably rarely occur in astronomy.)

In this case use can be made of the polar triangle but the geometrical significance of this method can best be illustrated by supposing the tangent planes to the sphere constructed at ABC the vertices of the original spherical triangle as in Fig. 5, these planes intersecting in the three lines TD, TE, TF and BD and DC being equal tangents to the great circle BC. It is obvious that angles such as TDB and TDC are right angles, and thus ETF is the supplement of EAF or the angle A of the original triangle; moreover the angle BDC is the dihedral angle between the planes TDBF and TDCE, and is ovbiously the supplement of the side BC of the original triangle. The trihedral angle at T thus determines a spherical triangle the sides and angles of which are the supplements of the angles and sides of the original triangle and this is therefore the polar triangle required.

Case VII. Given the simultaneous altitudes of two known stars to find the latitude.

Use the notation Z for zenith, P for pole and S, T for the two stars. If the angular distance between the stars is not already given let it be found by the construction of Case II. Let d_1 and d_2 be the N. P. D's of the stars, z_1 and z_2 their zenith distances and s the angular distance between the stars. The following construction depends on constructing the plane quadrilateral in the tangent plane to the celestial sphere through one of the stars S.

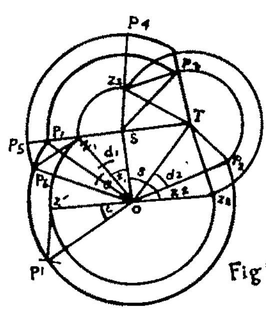

Fig 6.

Draw a line OS and draw TSP_1 at right angles to it. Make $\angle SOT = s$, $\angle SOZ_1 = z_1$, $SOP_1 = d_1$ $TOZ_2 = z_2$, $TOP_2 = d_2$, $OZ_2 = OZ_1$ $OP_2 = OP_1$ construct the triangle STP_3 having $SP_3 = SP_1$ and $TP_3 = TP_2$, and construct the triangle SZ_3T having $SZ_3 = SZ_1$ and $TZ_3 = TZ_2$. Then STP_3Z_3 is the projection of the figure found by the stars zenith and polar on the tangent plane through S. Construct anywhere a triangle Z′OP′ having its sides respectively equal to OZ_1, OP_1 and Z_3P_3, then Z′OP′ is the required colatitude. If the constructions be

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ON THE THEORY OF THE ABNORMAL ZEEMAN-EFFECT

BY

PANCHANON DAS, M.Sc.

[*Read April 17th, 1921.*]

When a line spectrum is examined with the source placed in a strong magnetic field, the line, when observed in a direction at right angles to the magnetic field, splits itself into three lines, of which the central line occupies the original position and the two outer components are equidistant from it. If it is observed in the direction of the field, the line splits itself into a doublet. This phenomenon is known as the Zeeman-effect. But in a large number of lines, the magnetic resolution is not so simple ; each line may be resolved into 4, 6, 8, 9 or even 13 components and these are not always symmetrically placed relative to the original line. The theory of normal Zeeman-effect was given by H. A. Lorentz on the classical electromagnetic theory, but this threw no light on the complicated Zeeman-effect. Sommerfeld[1] in 1916 utilised the atomic model of Bohr[2], and successfully applied the Quantum-theory to account for the normal Zeeman-effect, but even this had no marked advantage over the classical theory, in so far as Zeeman-effect was concerned. Sommerfeld's paper is open to some criticism as he neglected the squares of the magnetic field. We shall show that these terms are responsible for the assymmetry observed by Gmelin[2] and Dufour[3]. Secondly, we shall describe an atomic model, which, though somewhat speculative in character, satisfactorily explains Runge's law of multiple resolution which may be stated as follows :—

The distances of the lines from the central component are exact multiples of an aliquot part of the normal resolution.

Thirdly, we shall show that an extension of the dynamics of the $+H_2$ ion, leads to the result, that a part of the secondary spectrum of hydrogen should not show Zeeman-effect.

Lastly we shall investigate the dynamical features of a hydrogen atom in a magnetic field, taking relativity into account.

A study of the intensity and polarisation will be given in the next paper.

[1] Phil. Mag., 1913-15. [2] Physik. Zeitschrifte, 1908.
[3] Comptes Rendus, 1910.

The investigations of Sommerfeld were confined to the comparatively simple atom of hydrogen. The atom under our consideration has an inner ring of electrons and an external valency-electron, which takes part in emitting the visible radiations, all lying tn the same plane. Let the magnetic field H be perpendicular to this plane. We take the direction of H as the Z-axis of a system of axes rotating about the same, and in the plane of the ring take polar coordinates r, θ, of the valency-electron referred to moving axes. Let the prime-vector make an angle ϕ with a fixed direction at any instant.

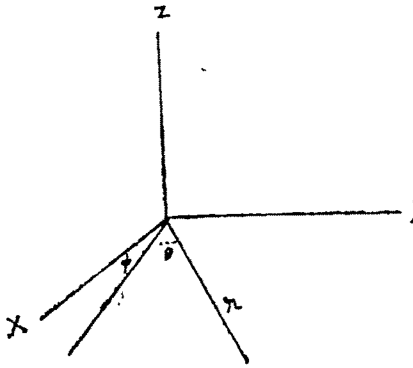


Fig. 1.

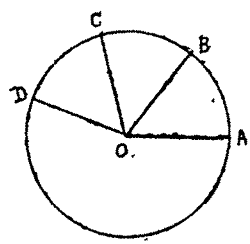


Fig. 2.

We assume that the nucleus has a charge $+Ze$, and that $Z-1$ negative electrons are symmetrically arranged in a ring of radius a, which is small compared with r. If we regard the charge $-(Z-1)e$ as uniformly distributed over the ring, the potential of the ring at any point in the plane is

$$= \frac{(Z-1)e^2}{2\pi}\int_0^{2\pi} \frac{d\psi}{\sqrt{r^2+a^2-2ar\cos\psi}}$$

$$=(Z-1)e^2\left[\frac{1}{r}+\frac{1}{4}\cdot\frac{a^2}{r^3}+\frac{9}{64}\cdot\frac{a^4}{r^5}\cdots\right].$$

The potential of the nucleus is $-\frac{Ze^2}{r}$.

Hence the potential under which the valency electron moves, is

$$V_1=-\frac{Ze^2}{r}+(Z-1)e^2\left[\frac{1}{r}+\frac{1}{4}\frac{a^2}{r^3}\cdots\right]$$

$$=-\frac{e^2}{r}+\frac{(Z-1)e^2a^2}{4r^3}+\cdots$$

The equations of motion of the valancy-electron are

$$m\{\ddot{r}-r(\dot{\theta}+\dot{\phi})^2\}=-\frac{\partial V_1}{\partial r}-\frac{eH}{c}r(\dot{\theta}+\dot{\phi}).$$

$$\frac{m}{r}\frac{d}{dt}\{r^2(\dot{\theta}+\dot{\phi})\}=\frac{eH}{c}\dot{r}.$$

If we put $\dot{\phi}=\frac{f}{2}$, where $f=\frac{e}{m}\cdot\frac{H}{c}$, these reduce to

$$r^2\dot{\theta}=\text{cost, and } m(\ddot{r}-r\dot{\theta}^2)=-\frac{\partial V_1}{\partial r}-mr\cdot\frac{f^2}{4}.$$

These are the equations of motion of a particle moving in a field of potential,

$$V=V_1+\frac{mf^2}{8}r^2.$$

We now proceed to the Hamilton—Jacobian form of the equation of motion.

If T be the kinetic energy the function S defined by

$S=2\int_0^t Tdt$ is Jacobi's Action-function. The generalised components of momenta are then $p_r=\frac{\partial T}{\partial q_r}=\frac{\partial S}{\partial q_r}$.

The energy-equation T+V=total energy=W may be expressed in terms of p_r, q_r etc. and W. If we replace p_r with $\frac{\partial S}{\partial q_r}$ we get the Hamilton-Jacobian partial differential equation. Thus,

$$T=\tfrac{1}{2}m(\dot{r}^2+r\dot{\theta}^2)=\frac{1}{2m}\left(p_r^2+\frac{p_\theta^2}{r^2}\right).$$

And $V=-\frac{e^2}{r}+\frac{(Z-1)e^2a^2}{4r^3}+\frac{mf^2r^2}{8}$ neglecting terms of higher order than a^2. Hence

$$\left(\frac{\partial S}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial S}{\partial \theta}\right)+2m\left[-\frac{e^2}{r}+\frac{(Z-1)e^2a^2}{4r^3}+\frac{mf^2r^2}{8}\right]=2mW.$$

This shows that θ is a cyclic coordinate, and we get $\dfrac{\partial S}{\partial \theta} = p_\theta = \text{const.}$

The angular Quantum-condition is

$$\int_0^{2\pi} p_\theta d\theta = nh \quad \text{or} \quad p_\theta = \frac{nh}{2\pi}$$

$$\therefore p_r^2 = \left(\frac{\partial S}{\partial r} \right)^2 = 2mW + \frac{2me^2}{r} - \frac{p_\theta^2}{r^2} - \frac{m^2 f^2}{4} r^2 - \frac{m(Z-1)e^2 a^2}{2r^3}.$$

or $$p_r = \sqrt{A + \frac{2B}{r} + \frac{C}{r^2} + \frac{D_1}{r^3} + D_2 r^2}$$

where $A = 2mW, \quad B = me^2, \quad C = -p_\theta^2,$

$$D_1 = -\frac{m(Z-1)e^2 a^2}{2}, \quad D_2 = -\frac{m^2 f^2}{4}.$$

The radial Quantum-condition is $\int p_r dr = n'h$.

According to Sommerfeld the Planck-constant h is the periodicity-modulus of the complex p_r arising from a closed integration including all the branch-points of the function p_r. Since the terms involving D_1, D_2 are small the branching-points are approximately the roots of $Ar^2 + 2Br + C = 0$. These roots are real, positive and unequal. The value of the integral is thus the sum of the residues at the two poles $r=0$ and $r=\infty$. We may write,

$$\int p_r dr = \int \sqrt{A + \frac{2B}{r} + \frac{C}{r^2}}\, dr + \int \left(A + \frac{2B}{r} + \frac{C}{r^2} \right)^{-\frac{1}{2}} \frac{D_1}{r^3}\, dr$$
$$+ \int \left(A + \frac{2B}{r} + \frac{C}{r^2} \right)^{-\frac{1}{2}} \frac{D_2}{2} r^2 dr$$

$$= 2\pi i \left[\frac{B}{\sqrt{A}} - \sqrt{C} + \frac{D_1 B}{2C\sqrt{C}} - \frac{D_2 B}{4\sqrt{A}\cdot A^2} \left(3C - \frac{5B^2}{A} \right) \right].$$

From this it follows that

$$W = -\frac{Nh}{(n+n'+a)^2} - \frac{f^2 h}{128\pi^2 N} \{3n^2(n+n'+a)^2 - 5(n+n'+a)^4\},$$

where a is a constant, and $N = \dfrac{2\pi^2 me^4}{h^3}$.

This is the total energy relative to rotating axes. We shall now find the total energy W′ relative to fixed axes.

$$W' = \frac{m}{2}\{\dot{r}^2 + r^2(\dot{\theta} - \dot{\phi})^2\} + V_1$$

$$= \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{m}{2}\cdot r^2\dot{\theta}\cdot f + V_1 + \frac{mf^2r^2}{8}$$

$$= \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2) + V + p_\theta\cdot\frac{f}{2} = W + \frac{f}{2}\cdot p_\theta,$$

or $$W' = -\frac{Nh}{(n+n'+a)^2} + \frac{f}{4\pi}nh - \frac{f^2h}{128\pi^2}\,\frac{1}{N}F(n,n'),$$

where $F(n,n') = 3n^2(n+n'+a)^2 - 5(n+n'+a)^4$.

If W_0 be the total energy without the magnetic field,

$$W' = W_0 + \frac{f}{4\pi}nh - \frac{f^2h}{128\pi^2N}F(n,n'),$$

or $$\Delta W = \frac{f}{4\pi}nh - \frac{f^2h}{128\pi^2N}F(n,n').$$

If $\Delta W'$ be the value corresponding to another configuration, of which the Quantum-numbers are p,p', then

$$h\Delta v = \Delta W' - \Delta W = \frac{f}{4\pi}(p-n)h - \frac{f^2h}{128\pi^2N}[F(p,p') - F(n,n')],$$

or $$\Delta v = -c\frac{d\lambda}{\lambda^2} = \frac{f}{4\pi}(p-n) - \frac{f^2}{128\pi^2N}[F(p,p') - F(n,n')],$$

where $f = \frac{e}{m}\cdot\frac{H}{c}$.

If we measure e in electromagnetic units, and write $N_0 = \frac{N}{c}$ =Rydberg's constant, we get

$$\frac{d\lambda}{\lambda^2} = \frac{e}{m}\cdot\frac{H}{c}\,\frac{n-p}{4\pi} - \frac{1}{8}\left(\frac{1}{4\pi}\cdot\frac{e}{m}\cdot\frac{H}{c}\right)^2\frac{1}{N_0}[F(n,n') - F(p,p')] \quad \ldots \quad (1)$$

By Sommerfeld's selective principle, $n-p=0$ or ± 1. Thus the second term shows that we have an assymmetry proportional to the square of field-strength. It is also obvious that this assymmetry

is of different magnitude in different lines. The following data are from Gmellin's experiments on the yellow line of mercury $\lambda = 5790$:—

H Gauss.	$d\lambda_e$ (A. U.)	$\frac{d\lambda}{H^2} 10^{20}$
9040	·0033	40·3
14450	·0091	43·6
16250	·011	43·3
20830	·018	41·8
26890	·021	43·9

Here $d\lambda_e$ is the increase in wave-length of the central-line.

The quotient $\frac{d\lambda_e}{H^2}$ is fairly constant. Substituting these values in formula (I), we find that the number $(n+n')$ must be between 8 and 10. Thus although the line does not belong to any particular series, yet assuming that it does, we find that its quantum-number is of the right magnitude.

Dufour has observed a kind of assymmetry in which the side-components are slightly shifted from their calculated positions. He finds this shift to be proportional to the square of the magnetic field-strength, which also follows from our formula.

If we plot $\Delta\nu$ against H we get a parabolic arc. This differs slightly from Voigt's experimental result. He found that the arc was hyperbolic, which asymptotically approached the straight line corresponding to normal Zeeman-effect.

§. 2.

If s electrons are arranged in a ring round the positive nucleus, then a state of motion is possible,* such that the angular distance between these electrons remain constant and the variable radii vectores from the nucleus to the electrons are equal to one another at the same instant. Thus the circle-locus on which they lie at any instant is either contracting or expanding. Let O be the nucleus and A, B, C etc. the negative electrons. Then $\angle AOB = \angle BOC = \angle COD$ etc. $= \alpha$, for all

* Cf. Sommerfeld "Atombau und Rontgen-Spektren." Physik. Zeits. Bd. 19, 1918.

time, and OA=OB=OC etc.$=r$, at any instant t. If the number of electrons is s, the angle $\alpha=\frac{2\pi}{s}$. The number of coordinates, which specify the position of such a system is only two, *viz.*, r and the angle θ, which any radius vector say OA, makes with a fixed direction. The potential of any electron is

$$-\frac{se^2}{r}+\frac{e^2}{r}\left[\frac{1}{\sin\frac{\alpha}{2}}+\frac{1}{\sin\frac{3\alpha}{2}}+\dots\frac{1}{\sin\frac{2s-1}{2}\alpha}\right]=-\frac{se^2}{r}+\frac{\mathrm{L}}{r}\text{ say.}$$

The equations of motion of the pth electron are

$$m\left[\ddot{r}-r\left\{\frac{d}{dt}(\theta+\overline{p-1}\alpha)\right\}^2\right]=-\frac{se^2}{r^2}+\frac{\mathrm{L}}{r^2},$$

and $$\frac{m}{r}\frac{d}{dt}[r^2(\theta+\overline{p-1}\alpha)]=0,$$

or $$m(\ddot{r}-r\dot{\theta}^2)=-\frac{se^2}{r^2}+\frac{\mathrm{L}}{r^2}.$$

and $$r^2\dot{\theta}=\text{const.}$$

Thus the motion of each of them is an ellipse with the positive nucleus at one focus. The equation to the path of the pth electron is

$$\frac{l}{r}=1+\epsilon\cos(\theta+\overline{p-1}\alpha).$$

When the magnetic field is brought in, we should expect that each of these ellipses will acquire a precessional movement. So we take moving axes such that the prime vector makes an angle ϕ with a fixed direction at any instant t.

The equations of motion of the electron, say A, become

$$m[\ddot{r}-r(\dot{\theta}+\dot{\phi})^2]=-\frac{se^2}{r^2}+\frac{\mathrm{L}}{r^2}-\frac{e\mathrm{H}}{c}r(\dot{\theta}+\dot{\phi})$$

and $$\frac{m}{n}\frac{d}{dt}\{r^2(\dot{\theta}+\dot{\phi})\}=\frac{e\mathrm{H}}{c}\dot{r}.$$

As before, put $\dot{\phi}=\frac{f}{2}$, where $f=\frac{e}{m}\cdot\frac{\mathrm{H}}{c}$. Thus,

$$m(\ddot{r}-r\dot{\theta}^2)=-\frac{se^2}{r^2}+\frac{\mathrm{L}}{r^2}-\frac{mf^2r}{4}\text{ and }r^2\dot{\theta}=\text{const.}$$

These are the equations of motion of a particle under a potential

$$-\frac{se^2}{r}+\frac{\mathrm{L}}{r}+\frac{mf^2r^2}{8}.$$

We shall see that the square of the field-strength does not take part in the multiple resolution, so we neglect $\frac{mf^2r^2}{8}$.

The kinetic energy T of the system is

$$T=\frac{sm}{2}(\dot{r}^2+r^2\dot{\theta}^2).$$

And the potential energy is

$$V=-\frac{s^2e^2}{r}+\frac{s}{2}\cdot\frac{L}{r}.$$

Then $p_r=\frac{\partial T}{\partial \dot{r}}=sm\dot{r}$, and $p_\theta=\frac{\partial T}{\partial \dot{\theta}}=smr^2\dot{\theta}$.

$\therefore$ the Hamilton-Jacobian equation is $T+V=W$

or
$$\frac{1}{2sm}\left[p_r^2+\frac{p_\theta^2}{r^2}\right]+\frac{s}{2}\,\frac{L}{r}=W.$$

From Quantum-conditions $p_\theta=\frac{nh}{2\pi}$ and $\int p_r dr=n_1 h$.

Let W′ be the absolute value of total energy

$$W'=\frac{sm}{2}\left[\dot{r}^2+r^2(\dot{r}+\dot{\phi})^2\right]+\frac{s}{2}\cdot\frac{L}{r}$$

$$=W\frac{sm}{2}\left[r^2+r^2\dot{\theta}^2\right]+p_\theta\cdot\frac{f}{2}+\frac{s}{2}\,\frac{L}{r}$$

$$=W+p_\theta\cdot\frac{f}{2}.$$

$$\therefore \Delta W=p_\theta\cdot\frac{f}{2}=\frac{1}{4\pi}\cdot\frac{e}{m}\cdot\frac{H}{c}nh.$$

If we make the hypothesis that when a homogeneous radiation takes place, all the electrons jump together into another configuration, the number of Quanta of energy emitted by them must be at least $sh\nu$.[1]

Thus
$$\Delta W_1-\Delta W_2=s\cdot h\Delta\nu=\frac{1}{4\pi}\,\frac{e}{m}\cdot\frac{H}{c}\,h(n-p)$$

or
$$\Delta\nu=\frac{1}{4\pi}\cdot\frac{e}{m}\cdot\frac{H}{c}\cdot\frac{n-p}{s}.$$

Sommerfeld's Selective principle, which states that $n-p=\pm 1$ or 0, is known to be violated, in may lines.[2] For the kind of atom we are

[1] Compare Wolfke's Viellinieon Specktrum de Wasserstoff Physik-Zeitsolite, 1920.

[2] Ef. Franck und Reiche, Zeito-f-Phys. Bd. 2, S. 159, footnote, 1920.

considering, we discard the principle altogether and lay down that n, p may have any values. The foregoing formula is then easily interpreted as Runge's law.

If the term involving the square of the magnetic field be taken into account, which is easily done, the multiple resolution will also show an assymmetry. Again, the calculation of total energy W_s without the magnetic field furnishes us with a new serial formula, which may represent many lines, for which any serial relation has hitherto been unknown. We find that

$$W_s = -N \cdot s^3 \cdot h\,(1-S_s)^2 \frac{1}{(n+n')^2}$$

where

$$S_s = \frac{1}{\sin\frac{a}{2}} + \frac{1}{\sin\frac{3a}{2}} + \dots \frac{1}{\sin\frac{2s-1}{2}a}.$$

This gives $v = N \cdot s^2 (1-S_s)^2 \left[\frac{1}{(m+m')^2} - \frac{1}{(n+n')^2} \right].$

This is similar to Balmer's formula in form, but the Rydbierg-constant N has a multiplying factor depending on the number of outer electrons.

§. 3.

We have already referred to a paper by Wolfke, in which be finds a series formula for the Schuman region of the secondary spectrum of hydrogen. He starts from Stark's[1] statement that the secondary spectrum of hydrogen or at least a part of it is emitted by the positively charged hydrogen molecule. Wolfke's model of such a molecule consists of a negative electron at rest and two positive nuclei describing the same circle about the electron. In conformity with the generalisation of the motion of such a system, we may regard the positive nuclei as describing ellipses with the electron at focus, while they lie on the same circle at any instant t. Thus another Quantum-number, *viz.*, that corresponding to r, becomes available. The series formula has the form

$$v = K \left[\frac{1}{(m+m')^3} - \frac{1}{(n+n')^3} \right],$$

and the Zeeman-effect is given by

$$(2) \quad \dots \quad \Delta v = \frac{1}{4\pi} \cdot \frac{e}{m} \cdot \frac{H}{c} (m-n).$$

[1] Atombau und Spektrallinien.

where M is the mass of a positive nucleus. Since we have now the additional numbers m', n', we can give any positive integral values to $m+m'$. and $n+n'$, while we put $m-n=\pm 1$ or 0. Thus we get a formula for Zeeman-effect consistent with the series formula. Now M is nearly thousand times as large as the mass of a negative-electron. Hence the magnetic-resolution of such lines is only $\frac{1}{1000}$th part of a normal resolution, and therefore cannot be measured by means of the instruments available at present. Hence these lines apparently would not show Zeeman effect at all. It is well-known that some lines in the secondary spectrum of hydrogen do not show any magnetic resolution.

§. 4.

The occurence of doublets and triplets of constant frequency-difference in ordinary lines, was shown by Sommerfeld to be a relativity effect. The gradual overlapping of the lines of a doublet into a single line in the presence of a magnetic field is known as the Paschen and Back[1] effect. Sommerfeld undertook to add a relativity-correction to the Zeeman-effect of hydrogen in order to explain this. But as he foud that the relativity-correction did not point to such an overlapping, he did not publish an account of his method. Recently Bohr calculated the relativity-correction by the method of planetary perturbations. He is said to have explained the distribution of intensity satisfactorily. By a different method we shall investigate the dynamics of an electron-system in the presence of a magnetic field and adopt the principle of relativity in so far as it regards mass as a function of velocity. As Sommerfeld has done we lay down as axioms that,

$$\frac{d}{dt}(\text{momentum}) = \text{impressed force},$$

and assume that the ordinary laws of vector-algebra hold good.

As before, we take a fixed set of axes (OX, OY, OZ) and a rotating prime vector which makes an angle ϕ with the X—axis. The equations of motion of the electron is

$$\frac{d}{dt}(m\dot{x}) = -\frac{e^2}{r^2}\cdot\frac{x}{r} - \frac{e\mathrm{H}}{c}\cdot\dot{y} \qquad \ldots \quad (1)$$

$$\frac{d}{dt}(m\dot{y}) = -\frac{e^2}{r^2}\cdot\frac{y}{r} + \frac{e\mathrm{H}}{c}\cdot\dot{x} \qquad \ldots \quad (2)$$

[1] Annalen der Physik, Bd. 39, 1912.

Multiplying (1) by $\frac{x}{r}$ and (2) by $\frac{y}{r}$ and anding them, we get,

$$\frac{d}{dt}(m\dot{r})-mr(\dot{\theta}+\dot{\phi})^2=-\frac{e^2}{r^2}-\frac{e\mathrm{H}}{e}r(\dot{\theta}+\dot{\phi}).$$

Similarly, multiplying (1) by $\frac{y}{r}$ and (2) by $\frac{x}{r}$ and taking the difference we get,

$$\frac{1}{r}\frac{d}{dt}\{mr^2(\dot{\theta}+\dot{\phi})\}=\frac{e\mathrm{H}}{c}\dot{r}.$$

If we put $\dot{\phi}=\frac{f}{r}$, where $f=\frac{e}{m}\cdot\frac{\mathrm{H}}{c}$, these again reduce to

$$r^2\dot{\theta}=\text{constant}.$$

and
$$\frac{d}{dt}(m\dot{r})-mr\dot{\theta}^2=-\frac{e^2}{r^2}-\frac{mf^2r}{4}.$$

As m is variable $\dot{\phi}$ is not constant, and there is no potential function. Hence Hamilton-Jacobian equations of motion cannot be formed.

If we neglect the term mf^2r, a potential exists however and a computation of total energy in terms of Quantum-numbers becomes possible. For kinetic energy we employ the relativiestic value $(m-m_0)c^2$. Thus the energy equation is

$$m_0c^2\left(\frac{1}{\sqrt{1-\beta^2}}-1\right)-\frac{e^2}{r}=\mathrm{W},$$

where $\beta^2=\frac{v^2}{c^2}$.

$$\therefore\frac{1}{\sqrt{1-\beta^2}}=1+\frac{1}{m_0c^2}\left(\mathrm{W}+\frac{e^2}{r}\right)$$

And
$$\beta^2=\frac{1}{c^2}[\dot{r}^2+r^2(\dot{\theta}+\dot{\phi})^2]$$

$$=\frac{1}{c^2m^2}\left[p_r^2+\frac{p_\theta^2}{r^2}+wp_\theta\right]$$

where $p_\theta=mr^2\dot{\theta}$ and $w\,.\,\frac{e\mathrm{H}}{c}$.

We have, $\int_0^{2\pi}p_\theta d\theta=nh$ or $p_\theta=\frac{nh}{2\pi}$.

Then we get

$$\frac{1}{1-\beta^2}=1+\frac{1}{c^2m_0^2}\left(p_r^2+\frac{p_\theta^2}{r^2}+wp_\theta\right)$$

$$=\left\{1+\frac{1}{m_0c^2}\left(W+\frac{e^2}{r}\right)\right\}^2.$$

$$\therefore p_r^2=A+\frac{2B}{r}+\frac{C}{r^2},$$

where $A=2m_0W+\frac{W}{c^2}-wp_\theta,\ B=m_0e^2+\frac{We^2}{c^2},$

$$C=\frac{e^2}{c^2}-p_\theta^2.$$

$$\therefore\ nh=\int p_r dr=2\pi 2\left(\frac{B}{\sqrt{A}}-C\right),$$

where $1+\frac{W}{m_0c^2}=\left(1+\frac{wp_\theta}{m_0^2c^2}\right)^{\frac{1}{2}}\left\{1+\frac{a^2}{(n'+\sqrt{n^2-a^2})}\right\}^{-\frac{1}{2}}$

where $a=\frac{2\pi e^2}{hc}.$

$$\therefore\ W=-\frac{1}{2}\cdot\frac{a^2m_0c^2}{(n+n')^2}+\frac{1}{2}\cdot\frac{wp_\theta}{m_0}\left\{1-\frac{1}{2}\cdot\frac{a^2}{(n+n')^2}\right\}$$

$$-\frac{1}{2}\cdot\frac{m_0c^2a^4}{(n+n')^4}\left(\frac{1}{4}+\frac{n'}{n}\right).$$

Thus, the first term represents the series term, the second the Zeeman-effect and the third, the fine structure of the lines.

My thanks are due to Prof. C. V. Raman for the helpful interest he has taken in this paper.

11

ON THE THEORY OF CONTINUED FRACTION

[*Fourth Paper*]

BY

HARIPADA DATTA, M.A. (EDIN.).

[*Read January 16th, 1921.*]

I. Let $a_1 + a_2 x + a_3 x^2 + \ldots$ ... (1)

$$= \frac{a_1}{1+}\,\frac{a_2 x}{1+}\,\frac{a_3 x}{1+}\ldots \qquad \ldots (2)$$

The $2n$th convergent $= \dfrac{\text{polynomial in } x \text{ and of degree } (n-1)}{\text{polynomial in } x \text{ and of degree } n}$,

and $(2n+1)$th convergent $= \dfrac{\text{polynomial of degree } n}{\text{polynomial of degree } n}$.

If the n convergent is converted into a power-series in x, then the first n terms of this series will coincide term for term with the first n terms of the series (1).*

The sixth convergent is of the form

$$\frac{\gamma_0 + \gamma_1 x + \gamma_2 x^2}{1 + \beta_1 x + \beta_2 x^2 + \beta_3 x^3}$$

and $= a_1 + a_2 x + a_3 x^2 + a_4 x^3 + a_5 x^4 + a_6 x^5 + Ax^6 + \ldots$

Now multiplying both sides by the denominator of the convergent and equating to zero the coefficients of x^3, x^4 and x^5, we obtain

$$a_1\beta_3 + a_2\beta_2 + a_3\beta_1 + a_4 = 0$$
$$a_2\beta_3 + a_3\beta_2 + a_4\beta_1 + a_5 = 0$$
$$a_3\beta_3 + a_4\beta_2 + a_5\beta_1 + a_6 = 0$$

$$\therefore \beta_3 = -\frac{{}^1K_3}{K_3}, \quad \beta_2 = \frac{{}^2K_3}{K_3}, \quad \beta_1 = -\frac{{}^3K_3}{K_3}.$$

Where ${}^1K_3$, ${}^2K_3$, ${}^3K_3$ and K_3 are determinants and obtained by deleting the 1st, 2nd, 3rd and the 4th column of the array

$$\begin{Vmatrix} a_1 & a_2 & a_3 & a_4 \\ a_2 & a_3 & a_4 & a_5 \\ a_3 & a_4 & a_5 & a_6 \end{Vmatrix}.$$

* This was first shown by Stein.

As $\quad f_8=(a_8x+a_7x+1)f_6-a_6a_7\lambda^2 f_4 \quad \ldots \quad (3)$

so, by equating the coefficients of both sides of (3) we get

$$I=1$$

$$-\frac{{}^4K_4}{K_4}=-\frac{{}^3K_3}{K_3}+a_7+a_8$$

$$\frac{{}^3K_4}{K_4}=\frac{{}^2K_3}{K_3}-a_7\frac{{}^3K_3}{K_3}-a_8\frac{{}^3K_3}{K_3}-a_6a_7$$

$$-\frac{{}^2K_4}{K_4}=-\frac{{}^1K_3}{K_3}+a_7\frac{{}^2K_3}{K_3}+a_8\frac{{}^2K_3}{K_3}+a_6a_7\frac{{}^2K_2}{K_2}$$

$$\frac{{}^1K_4}{K_4}=0-a_7\frac{{}^1K_3}{K_3}-a_8\frac{{}^1K_3}{K_3}-a_6a_7\frac{{}^1K_2}{K_2}.$$

Multiplying the rows by a_7, a_6, a_5, a_4 and a_3 in order and adding up vertically we obtain

$$0=\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ a_2 & a_3 & a_4 & a_5 \\ a_3 & a_4 & a_5 & a_6 \\ a_4 & a_5 & a_6 & a_7 \end{vmatrix} \div K_3-a_6a_7\begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix}.$$

$$\therefore a_6a_7=\frac{K_4}{K_3}\cdot\frac{K_2}{K_3}.$$

Thus we can show that

$$a_{2n}a_{2n+1}=\frac{K_{n-1}K_{n+1}}{K_n^2} \quad \ldots \quad (4)$$

Again $\quad f_7=1-\frac{{}^3P_3}{P_3}x+\frac{{}^2P_3}{P_3}x^2-\frac{{}^1P_3}{P_3}x^3.$

Where P's are determinants of order 3 and obtained from

$$\begin{Vmatrix} a_2 & a_3 & a_4 & a_5 \\ a_3 & a_4 & a_5 & a_6 \\ a_4 & a_5 & a_6 & a_7 \end{Vmatrix}$$

And $\quad f_9=(a_9x+a_8x+1)f_7-a_7a_8x^2f_5.$

Hence in the similar way we can show that

$$a_7a_8=\frac{P_2P_4}{P_3^2}$$

and $$a_{2n+1}a_{2n+2}=\frac{P_{n-1}P_{n+1}}{P_n^2} \qquad \dots \quad (5)$$

From (4) and (5) we have

$$\left.\begin{aligned} a_{2n}&=-\frac{K_{n-1}}{K_n}\cdot\frac{P_n}{P_{n-1}} \\ a_{2n+1}&=-\frac{K_{n-1}}{K_n}\cdot\frac{P_{n-1}}{P_n} \end{aligned}\right\} \qquad \dots \quad (6)$$

The convergents are

$$\left.\begin{aligned} f_{2n}&=1-\frac{{}^nK_n}{K_n}x+\frac{{}^{n-1}K_n}{K_n}x^2-\dots+(-1)^n\frac{{}^1K_n}{K_n}x^n \\ f_{2n+2}&=1-\frac{{}^nP_n}{P_n}x+\dots+(-1)^n\frac{{}^1P_n}{P_n}x^n \end{aligned}\right\} \qquad \dots \quad (7)$$

II. Let us convert the series

$$q^{a_1}+q^{a_2}x+q^{a_3}x^2+\dots$$

into the C. F. of the form (2), where

$$l=a_1-2a_2+a_3=a_2-2a_3+a_4=\dots \qquad \dots \quad (8)$$

and $$l\neq 0.$$

From (8)

$$a_p=\tfrac{1}{2}(p-3)(p-2)a_1-(p-3)(p-1)a_2+\tfrac{1}{2}(p-2)(p-1)a_3 \qquad \dots \quad (9)$$

(i) $$K_4=\begin{vmatrix} q^{a_1} & q^{a_2} & q^{a_3} & q^{a_1} \\ q^{a_2} & q^{a_3} & q^{a_4} & q^{a_5} \\ q^{a_3} & q^{a_4} & q^{a_5} & q^{a_6} \\ q^{a_4} & q^{a_5} & q^{a_6} & q^{a_7} \end{vmatrix}$$

$$=q^{a_4+a_5+a_6+a_7}\begin{vmatrix} q^{a_1-a_4} & q^{a_2-a_4} & q^{a_3-a_4} \\ q^{a_2-a_5} & q^{a_3-a_5} & q^{a_4-a_5} \\ q^{a_3-a_6} & q^{a_4-a_6} & q^{a_5-a_6} \\ q^{a_4-a_7} & q^{a_5-a_7} & q^{a_6-a_7} \end{vmatrix}$$

(by $\text{row}_1-\text{row}_2$, $\text{row}_2-\text{row}_3$, $\text{row}_3-\text{row}_4$).

$$=q^{a_7}\begin{vmatrix} q^{a_1}(1-q^{a_2-a_5-a_1+a_4}) & q^{a_2}(1-q^{a_3-a_5-a_2+a_4}) & q^{a_3}(1-q^{a_4-a_5-a_3+a_4}) \\ q^{a_2}(1-q^{a_3-a_6-a_2+a_5}) & q^{a_3}(1-q^{a_4-a_6-a_3+a_5}) & q^{a_4}(1-q^{a_5-a_6-a_4+a_5}) \\ q^{a_3}(1-q^{a_4-a_7-a_3+a_6}) & q^{a_4}(1-q^{a_5-a_7-a_4+a_6}) & q^{a_5}(1-q^{a_6-a_7-a_5+a_6}) \end{vmatrix}$$

which by (9) becomes

$$=q^{a_7}\left(1-q^{-3l}\right)\left(1-q^{-2l}\right)\left(1-q^{-l}\right)K_3.$$

Similarly

$$K_n=q^{a_{2n-1}}\left(1-q^{-l}\right)\left(1-q^{-2l}\right)\cdots\left(1-q^{-(n-1)l}\right)K_{n-1}$$

$${}^1K_n=q^{a_{2n}}\left(1-q^{-l}\right)\left(1-q^{-2l}\right)\cdots\left(1-q^{-(n-1)l}\right){}^1K_{n-1}$$

$${}^2K_n=q^{a_{2n}}\left(1-q^{-nl}\right)\left\{\left(1-q^{-l}\right)\left(1-q^{-2l}\right)\cdots\left(1-q^{-(n-2)l}\right)\right\}{}^2K_{n-1}$$

$${}^3K_n=q^{a_{2n}}\left(1-q^{-nl}\right)\left(1-q^{-(n-1)l}\right.\left\{\left(1-q^{-l}\right)\cdots\left(1-q^{-(n-\ )l}\right)\right\}{}^3K_{n-1}.$$

... ...

$${}^kK_n=q^{a_{2n}}\left\{\left(1-q^{-nl}\right)\left(1-q^{-(n-1)l}\right)\cdots\left(1-q^{-2l}\right)\right\}K_{n-1}.$$

If $\dfrac{(1-q^n)(1-q^{n-1})\dots(1-q^{n-r+1})}{(1-q^r)(1-q^{r-1})\dots(1-q)} = \begin{bmatrix} 1-q^n \\ r \end{bmatrix}$,

then,

$$\frac{{}^nK_n}{K_n} = q^{a_{2n}-a_{2n-1}} \begin{bmatrix} 1-q^{-nl} \\ 1 \end{bmatrix}$$

which by (9)

$$= q^{(2n-3)l-(a_3-a_2)} \begin{bmatrix} 1-q^{-nl} \\ 1 \end{bmatrix}$$

$$\frac{{}^{n-1}K_n}{K_n} = q^{(4n-2.4)l-3(a_3-a_2)} \begin{bmatrix} 1-q^{-nl} \\ 2 \end{bmatrix}$$

$$\frac{{}^{n-2}K_n}{K_n} = q^{(6n-3.5)l-3(a_3-a_2)} \begin{bmatrix} 1-q^{-nl} \\ 3 \end{bmatrix}$$

... ...

$$\frac{{}^2K_n}{K_n} = q^{\{2(n-1)n-(n-1)(n+1)\}l-(n-1)(a_2-a_3)} \begin{bmatrix} 1-q^{-nl} \\ n-1 \end{bmatrix}$$

$$\frac{{}^1K_n}{K_n} = q^{\{2n^2-n)n+2)\}l-n(a_3-a_2)}.$$

Therefore $\quad f_{2n} = 1 - \begin{bmatrix} 1-q^{-2l} \\ 1 \end{bmatrix} q^{(2n-3)l-(a_2-a_3)}x$

$$+\begin{bmatrix} 1-q^{-nl} \\ 2 \end{bmatrix} q^{(4n-2.4)l-2(a_2-a_3)}x^2 + \dots \dots \dots$$

$$\dots + (-1)^n q^{\{2n^2-n(n+2)\}l-n(a_2-a_3)} x^n. \qquad \dots \quad (10)$$

By increasing the suffixes of a's by unity, we obtain

$$f_{2n+1} = 1 - \begin{bmatrix} 1-q^{-nl} \\ 1 \end{bmatrix} q^{(2n-3)l-(a_3-a_4)}x\dots$$

$$\dots + (-1)^n q^{\{3n^2-n(n+1)\}.l-n(a_3-a_4)}x^n \qquad (11)$$

(*ii*) $\dfrac{P_n}{P_{n-1}} = q^{a_{2n}}\left(1-q^{-l}\right)\left(1-q^{-2l}\right)\dots\left(1-q^{-(n-1)l}\right)$

$$\left.\begin{aligned} a_{2n} &= -\frac{K_{n-1}}{K_n}\cdot\frac{P_n}{P_{n-1}} = -q^{a_{2n}-a_{2n+1}}, \\ a_{2n+1}{}^* &= -\frac{K_{n+1}}{K_n}\cdot\frac{P_{n-1}}{P_n} = -\left(1-q^{-nl}\right)q^{a_{2n+1}-a_{2n}} \end{aligned}\right\}$$

[* *For the contracted form of C. F. for the quotient of two series, see "On the failure of Heilermann's Theorem" Proc. Edin. Math. Soc. Vol. XXXV (Part 3) 1916-17*].

Ex. $\quad 1+q^{1}x+q^{4}x^{2}+q^{9}x^{3}+\ldots+q^{n^2}x^{n}+\ldots$

$$\left.\begin{array}{l} a_{2n}=-q^{4n+1} \\ a_{2n+1}=(1-q^{-2n})q^{4n+3} \\ l \text{ being } 2 \end{array}\right\} \text{Eissenstein}$$

Ex. 2. If $\quad a_m=\text{A}+\text{B}\{m+(m+1)+\ldots+(m+r-1)\}$
$\qquad\qquad+\text{C}\{m(m+1)+(m+1)(m+2)e+\ldots(m+r-1)(m+r)\}$

Then $\quad l=2r$

$$a_{2n}=-q^{\text{B}r+\text{C}r(4n+r-1)}$$

$$a_{2n+1}=-q^{\text{B}r+\text{C}r(4n+r+1)}\times\left(1-q^{-2rn}\right)$$

A, B, C, are constants.

Ex. 3. If $\quad a_m=m^2+(m+1)^2+\ldots+(m+r-1)^2,$

then $\quad l=2r$

$$\left.\begin{array}{l} a_{2n}=-q^{r(4n+r-2)} \\ a_{2n+1}=-q^{r(4n+r)}\left(1-q^{-2nr}\right) \end{array}\right\}$$

Ex. 4. The following known series satisfy the relation (8):

(*i*) $x+q^{4}x^{3}+q^{12}x^{5}+q^{24}x^{7}+\ldots+q^{n(2n+1)}x^{2n+1}+\ldots$
$\quad l=4$

(*ii*) $1+q^{2}x^{2}+q^{8}x^{4}+q^{18}x^{6}+\ldots+q^{2n^2}x^{2n}+\ldots$
$\quad l=4$

[Jacobi, Ges. Werke 1, page 230.]

This series is same as that of Ex 1.

12

On the Inverse of an undegenerate non-plural quadrate Slope.

BY

Sasindrachandra Dhar.

[*Read April 17th, 1921.*]

1. It is definitely known that the inverse of a simple slope is also a simple slope whose *non-zero* elements can be *uniquely* determined in succession; but the corresponding case of a compound slope haf not yet been completely investigated. The object of this paper is to show that the equation given by the skeleton matrices

$$\begin{array}{c} \\ \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_r \end{array}\begin{array}{c} \begin{array}{cccc} \alpha_1 & \alpha_2 & \dots & \alpha_r \end{array} \\ \left[\begin{array}{cccc} A_{11}, & A_{12}, & \dots & A_{1r} \\ A_{21}, & A_{22}, & \dots & A_{2r} \\ \dots & \dots & \dots & \dots \\ A_{r1}, & A_{r2}, & \dots & A_{rr} \end{array}\right] \end{array} \begin{array}{c} \begin{array}{cccc} \alpha_1 & \alpha_2 & \dots & \alpha_r \end{array} \\ \left[\begin{array}{cccc} X_{11}, & X_{12}, & \dots & X_{1r} \\ X_{21}, & X_{22}, & \dots & X_{2r} \\ \dots & \dots & \dots & \dots \\ X_{r1}, & X_{r2}, & \dots & X_{rr} \end{array}\right] \end{array} = \begin{array}{c} \begin{array}{cccc} \alpha_1 & \alpha_2 & \dots & \alpha_r \end{array} \\ \left[\begin{array}{cccc} I_{11}, & I_{12}, & \dots & I_{1r} \\ I_{21}, & I_{22}, & \dots & I_{2r} \\ \dots & \dots & \dots & \dots \\ I_{r1}, & I_{r2}, & \dots & I_{rr} \end{array}\right] \end{array} \quad (1)$$

$$i.e., \qquad AX = I$$

of the type $\{\pi, \pi\}\ \{\pi, \pi\} = \{\pi, \pi\}$ in which A is a given undegenerate *non-plural* quadrate slope, by which is meant a quadrate slope in which the index numbers are all different, and I, a unit matrix, has a unique solution in which X is also a non-plural quadrate slope and to find all the elements of X. We loose here no generality by supposing the index numbers $\alpha_1, \alpha_2, \dots \alpha_r$ which are all different, arranged in order of magnitude.

The equation determining the element common to the ith horizontal and the jth vertical rows of X_{uv} is that obtained by equating the product of the ith horizontal row of the uth horizontal minors of A and the jth vertical row of the vth vertical minors of X, to the element common to the ith horizontal row of the uth horizontal minors of I and

the jth vertical row of the vth vertical minors of I, *i.e.*, by taking the equation

$$[A_{u1}, A_{u2}, \ldots A_{ur}]\begin{bmatrix} X_{1v} \\ X_{2v} \\ \vdots \\ X_{rv} \end{bmatrix} = I_{uv} \qquad \ldots \quad (2)$$

i.e., $$T_1+T_2+\ldots+T_s+\ldots+T_r=I_{uv}, \qquad \ldots \quad (3)$$

where $T_s=A_{us}X_{sv}$

and equating the elements common to the ith horizontal and jth vertical row, that is

$$E(u, v\,;\, i, j)\equiv t_1+t_2+\ldots+t_s+\ldots+t_r=\gamma, \qquad \ldots \quad (4)$$

where $\gamma=0$, unless $v=u$ and $j=i$, in which case $\gamma=1$.

2. From the general theorem on *the evaluation of the product matrix*, of Prof. Cullis,* we find that

(*i*) When $[a]_m^p[b]_p^n=[x]_m^n$ *i.e.* $A.B=X$ is a product of *two* most general matrices having given basical nullities, and x_{ij} is a non-zero element of $[x]_m^n$, the non-zero terms in the expression

$$x_{ij}=\sum a_{iu}b_{uj} \qquad \ldots \quad (5)$$

are given by

$$\left.\begin{matrix} u \nless 1+a_2, & u \ngtr p-\beta_1 \\ u \nless i+a_0, & u \ngtr j-\beta_0 \end{matrix}\right\} \qquad \ldots \quad (6)$$

where a_0, β_0 are the least effective difference-weight; a_1, β_1 the horizontal nullities and a_2, β_2, the vertical nullities of A and B respectively.

(*ii*) When A is a general simple slope, and B a general simple matrix, this range becomes

$$u \nless i+a_0,\ u \ngtr p. \qquad \ldots \quad (7)$$

* Dr. C. E. Cullis, *Bull. Cal. Math. Soc.*, Vol. XI, No. 3, p. 147 (1920).

(*iii*) When A and B are both general simple slopes, the range becomes

$$u \nless i+a_0, \quad u \ngtr j-\beta_0. \quad \dots \quad (8)$$

From the above relations it would be easy to get the developed expression for (4), for t_s can be written as

$$t_s = \Sigma l_{ip_s} \lambda_{p_s j} \quad \dots \quad (6)$$

where l_{ip_s} is an element of A'_{s}, and $\lambda_{p_s j}$ of X_{s}, and the range of p_s can be easily obtained from above.

3. If we call the element common to the ith horizontal and the jth vertical rows of X_{uv} as x, the equation (4) can be written as

$$E(u, v\,;\, i, j) \equiv ax+\beta=\gamma,$$

where $\gamma=0$ or 1, $a \neq 0$ and β is a function (if all such equations are suitably arranged) of known quantities.

From an examination of the forms obtained in the proceeding article, it will be at once evident that

(*i*) *the expression* $E(u, v\,;\, i, j)$ *is isobaric in the difference-weights of the elements of* A *and* X, *having constant difference-weight* $(j-i)$.

(*ii*) *the coefficient* a *of* x *in the expression* $E(u, v\,;\, i, j)$ is always a *diagonal element of* A, *lying in the same horizontal line of it as* x *occupies in* X.

For, from the manner by which (3) is obtained, we find that if x be the element of X_{uv} common to the ith horizontal and jth vertical rows of it, then it occupies a position in the compound matrix X, obtained by taking the element common to the $(a_1+a_2+\dots+a_{u-1}+i)$th *horizontal* and $(a_1+a_2+\dots+a_{v-1}+j)$th *vertical* rows. Hence the element of A by which it is multiplied is obtained from the left-hand multiplier of (2) by taking the element common to the ith *horizonal* and $(a_1+a_2+\dots+a_{u-1}+i)$th *vertical* rows of it. This latter element, therefore, occupies a position in the compound matrix A, common to the $(a_1+a_2+\dots+a_{u-1}+i)$th *horizontal* and $(a_1+a_2+\dots+a_{u-1}+i)$th *vertical* raws and hence is a diagonal element of A.

4. The *necessary* and *sufficient* conditions that the compound matrix A of the given class shall be an undegenerate quadrate slope are that if k_{ij} be an element of a constituent matrix of A, *i.e.*, of $[k]^{\lambda}_{\mu}$, then

$$\left.\begin{array}{l} k_{ij}=0, \text{ if } j-i<\lambda-\mu, \text{ when } \lambda>\mu \\ \qquad\qquad \text{or} \\ k_{ij}=0, \text{ if } j-i<0, \qquad \text{when } \lambda \leq \mu \end{array}\right\} \quad \dots \quad (10)$$

Hence if we assume A to be an undegenerate quadrate slope, the equation (4) can be written, by the help of the relations (7) and (10) as

$$E(u, v: i, j) \equiv t_1 + t_2 + \ldots + t_s + \ldots + t_r = \gamma, \qquad \ldots \quad (11)$$

where $\gamma = 0$ or 1, and $t_s = \Sigma l_{ip_s} \lambda_{p_s j}$, for integral values of p_s given by

$$(i) \qquad (a_s - a_u) + i \leq p_s \leq a_s, \quad \text{when } s < u \qquad \ldots \quad (12)$$

$$(ii) \qquad i \leq p_s \leq a_s, \quad \text{when } u \leq s \leq r \qquad \ldots \quad (13)$$

5. We will now proceed to prove that *when* A *is a quadrate slope*, X *is also a quadrate slope of the same type as* A. For this purpose, let us proceed to find the elements of X in the following order :—

We determine in succession elements having negative difference-weights in the order

$$1 - a_1, 2 - a_1, \ldots 1 - a_2, 2 - a_2, \ldots 1 - a_r, 2 - a_r, \ldots -1,$$

with this procedure that in finding elements of difference-weight k (say) lying in a constituent matrix X_{uv}, we shall find them in the order

$$x_{k+1, 1}, \quad x_{k+2, 2}, \quad x_{k+3, 3}, \ldots$$

and further, that we shall find the elements of difference-weight $-k$ as they lie in parametric diagonal lines, beginning from the base and gradually proceeding to the apex.

If we adopt the above procedure, then for instance, in determining the element $x_{j+k, j}$ of difference weight $-k$, contained in X_{uv}, we will find that the expression (11) will contain only elements of difference weight less than $-k$ as well as elements of difference weight $-k$, which have been already determined.

The above will evidently hold, if we can show that the expression $E(u, v; i, j)$ can contain

(*i*) no elements of difference-weights greater than $(-k)$

(*ii*) no elements of difference weight $-k$ belonging to to the constituent matrices $X_{1v}, X_{2v}, \ldots X_{u-1, v}$;

but may contain

(*iii*) elements of difference-weight $(-k)$ belonging to $X_{u, v}, X_{u+1, v}, \ldots X_{r, v}$. ... (14)

If we now put $i = j + k$ in (11), (12) and (13) we get that

$$t_1 + t_2 + \ldots + t_s + \ldots + t_r = 0$$

where $t_s = \Sigma l_{j+k, p_s} \lambda_{p_s j}$, for integral values of p_s given by

$$a_s - a_u + j + k \leq p_s \leq a_s, \text{ when } s < u$$

$$j + k \leq p_s \leq a_s, \text{ when } u \leq s \leq r.$$

Therefore, $t_1, t_2, \ldots t_{u-1}$ cannot contain elements having their difference-weights

$$>j-p_s, \text{ where } p_t=\{(a_t-a_u)+j+k\}$$
$$i.e., \quad >-k-(a_s-a_u)$$
$$\geq -k;$$

but however, $t_u, t_{u+1}, \ldots t_r$ may contain elements having their difference-weights

$$\leq j-p_s, \text{ when } u \leq s \geq r \text{ and } p_s=j+ko$$
$$\leq -k.$$

Hence the relations (14) are proved.

In determining the element having the least difference-weight *viz.* $x_{a_1, 1}$, we find that it must be zero. Hence from above, elements of difference-weight $2-a_1$ must be zero and so also elements of difference-weight $3-a_1$, and so on. Hence generally, *all those elements of x, which have negative difference-weight, must be zero*, that is, if ξ_{ij} be an element of the constituent matrix $[\xi]^{\lambda}_{\mu}$ of X, then

$$\xi_{ij}=0, \text{ if } j-i<0. \qquad \ldots (15)$$

6. We will next prove that *the elements of difference-weight o, contained in constituent matrices having negative difference-weight, are all zero.** Here also we calculate the elements in succession, by adopting the same procedure as we have used in the previous article. Hence in calculating the element X_{ii} (say) contained in X_{uv} where $u>v$, we will show that $E(u, v; i, j)$ can contain no elements of difference-weight greater than zero but may contain elements of difference-weight less than zero and elements of difference weight zero, which has been already found. This is easily seen by considering (11), (12) and (13) when $j=i$, for then it will be seen that $t_1, t_2, \ldots t_{u-1}$ cannot contain elements having their difference-weights *i.e.*,

$$> i-p_s, \text{ when } s<u \text{ and } p_s=\{(a_s=a_u)+i\}$$
$$>i-\{(a_s-a_u)+i\}$$
$$i.e., \quad \geq 0 \qquad , \text{ for } (a_s-a_u) \text{ is pos.},$$

and further that $t_u, t_{u-1}, \ldots t_r$ may contain elements, having difference-weights

$$\leq i-p_s \text{ when } u \leq s \leq r \text{ and } p_s=i$$
$$\leq 0.$$

Hence from what we have proved in § 5 and also from the above property, it will be easily seen that these elements of difference-weight zero, must all vanish, *i.e.*, if ξ_{ij} be an element of the constituent

matrix $[\xi]^{\lambda}_{\mu}$ of X, where $\lambda > \mu$, then

$$\xi_{i\,j} = 0, \text{ if } j-i-0. \qquad \ldots \quad \ldots \quad (16).$$

7. We now proceed to determine elements of positive difference-weights, ranging from 1 upwards, lying in constituent matrices having negative difference-weights. In finding out elements of pos. difference-weights contained in $X_{u\,v}$ (say), where $u > v$, we shall determine its elements having difierence-weights from i up to k, where $k < a_v - a_u$.

Here also we determine in succession elements having difference-weights :—1, 2,...etc, in the same order as in § 5 and § 6— that is, we shall determine elements of a given difference-weight as they lie in parametric diagonal lines beginning with that line that lies nearest to the base and gradually going up to the leading diagonal line of X.

If we adopt the above procedure, we will now prove that in determining the element (say) $x_{i,\,i+k}$ of diff.-wt. k, contained in $X_{u\,v}$ where $u > v$ and $k < a_v - a_u$, the expression $E(u, v\, j\; i, j)$ cannot contain.

(*i*) elements of difference-weights greater than k,

(*ii*) elements of positive difference-weight contained in $X_{1\,v}, X_{2\,v}, X_{v,\,i}$,

(*iii*) elements of difference-weight k contained in $X_{i+1,\,i}, X_{v+2}, \ldots X_{u-1,\,v}$

but may contain

(*iv*) elements of difference-weights k contained in $X_{u,\,v}, X_{u+1},\ldots X_{r,\,v}$.

Now, examining the relations (11), (12) and (13) we get that $t_1, t_2, \ldots t_i$ cannot contain elements having their difference-weights

$$> i + k - p_s, \text{ where } p_s = (a_s - a_u) + i$$

$$\text{i.e., } > k - (a_s - a_u), \text{ which is a negative quantity.}$$

Also, $t_{v+1}, t_{i+2}, \ldots t_{u-1}$ cannot contain elements having difference-weights

$$\geq i + k - p_s, \text{ when } v \geq s \geq u \text{ and } p_s = (a_s - a_u) + i$$

$$\text{i.e., } > k - (a_s - a_u), \text{ where } v \leq s \leq u$$

$$\text{i.e., } \geq k.$$

And further, $t_u, t_{u+1}, \ldots t_r$ may contain elements having difference-weights

$$\leq i + k - p_s, \text{ when } u \leq s \leq r \text{ and } p_s = i,$$

$$\text{i.e., } \leq k.$$

Hence it will be easy to prove that all such elements are zero. *i.e.* if $\xi_{i,j}$ where $j>i$ be an element of $[\xi]^{\lambda}_{\mu}$ where $\lambda>\mu$, then

$$\xi_{i,j}=0 \quad \text{if } j-i<\lambda-\mu. \qquad \ldots \quad (17)$$

Thus from (15), (16), and (17) we have that if $[\xi]^{\lambda}_{\mu}$ be a constituent matrix of X, then

$$\left.\begin{array}{ll} \xi_{i,j}=0, \text{ if } j-i<\lambda-\mu, & \text{when } \lambda>\mu \\ \xi_{i,j}=0, \text{ if } j-i<0 & \text{when } \lambda<\mu. \end{array}\right\} \qquad \ldots \quad (18)$$

which are the *necessary* and *sufficient* conditions that X is a quadrate slope of the same type as A.

8. We have obtained above that the inverse of an undegenerate quadrate slope must also be an undegenerate quadrate slope and therefore, from (8) and (9), we get

$$E(u, v, i, j) \equiv t_1+t_2+\ldots+t_s+\ldots+t_r=\gamma,$$

where $t_s=\sum l_{s,i} p_s \lambda p_{s,j}$, for integral values of p_s given by

(*a*) when $u<v$,

$$\left.\begin{array}{lll} (i) & (a_s-a_u)+i\leq p_s\leq j, & \text{when } s<u \\ (ii) & i\leq p_s\leq j, & \ldots \ u\leq s\leq v \\ (iii) & i\leq p_s\leq j-(a_v-a_s)\ldots & v<s\leq r. \end{array}\right\} \qquad \ldots \quad (19)$$

(*b*) when $v<u$

$$\left.\begin{array}{lll} (i) & (a_s-a_u)+i\leq p_s\leq j, & \text{when } s\leq v \\ (ii) & (a_s-a_u)+i\leq p_s\leq j-(a_v-a_s, & \text{when } v<s<u \\ (iii) & i<p_s<j-(a_v-a_s), & \text{when } u\leq s\leq r \end{array}\right\} \ldots \quad (20)$$

(*c*) when $v=u$.

$$\left.\begin{array}{lll} (i) & (a_s-a_u)+i\leq p_s\leq j, & \text{when } s<u \\ (ii) & i\leq p_s\leq j, & \text{when } s=u \\ (iii) & i\leq p_s\leq j-(a_u-a_s), & \text{when } u\leq s\leq r. \end{array}\right\} \ldots \quad (21)$$

9. We will now proceed to find the other elements of X such that the relation (1) is satisfied. In doing so, we shall calculate successively by (19), (20), and (21), the elements having their difference-weights in the order 0, 1, 2,...etc.

In calculating elements of a given difference-weight $+k$ (say), we shall find them as they lie in parametric diagonal lines, beginning with the line nearest to the base and gradually going towards the apex, and

further, when finding elements of a given diagonal line of a constituent matrix, we shall calculate them in the order

$$x_{1,k+1}, x_{2,k+2}, x_{3,k+3}, \ldots$$

If we proceed to find the elements in the above way, we will prove that when finding the element $x_{i,i+k}$ (say) of X_{uv}, $E(u, v, i, j)$ cannot contain

(i) elements having difference-weights greater than k

(ii) elements of difference-weights k contained in

$$X_{1v}, X_{2v}, \ldots X_{u-1}, v.$$

but may contain

(iii) elements of difference-weights k contained in

$$X_{uv}, X_{u+1,v}, \ldots X_{rv}.$$

This can be at once seen by supposing $v<u$, $v=u$ or $v>u$ and considering the relations (19), (20) and (21) respectively. For instance, if we take $v>u$, then by (19) we will find that $t_1, t_2, \ldots t_{u-1}$ cannot contain elements having their difference-weights

$$>i+k-p_s, \text{ when } s<u \text{ and } p_s=(a_s-a_u)+i$$

$$i.e., \quad >k-(a_s-a_u)$$

$$i.e., \quad \geq k,$$

but t_u, $t_{u+1}, \ldots t_v$ have elements having difference-weights

$$\leq i+k-p_s, \text{ where } u\leq s\leq v \text{ and } p_s=i$$

$$\leq k,$$

and further t_{v+1}, $t_{v+2}, \ldots t_r$ can have elements having difference-weights

$$\leq i+k-p_s, \; v<s\leq r \text{ and } p_s=i$$

$$\leq k.$$

Hence $E(u, v, i, j)$ for determining $x_{i,\,i+k}$ can contain no elements which has not been already determined. Exactly the same thing may be proved when $x_{i,\,i+k}$ belongs to X_{uv} where $v\leq u$.

Now, in the above scheme of determining the elements of X, we shall have to start with elements of the leading diagonal line, whose values are evidently the *reciprocals* of the corresponding element of A and therefore *non-zero*. Therefore, all elements which shall find after it, will also be *non-zero* and it is easy to calculate them successively by following the procedure indicated above.

Before concluding, I should like to acknowledge my indebtedness to Dr. C. E Cullis at whose suggestion I took up the problem.

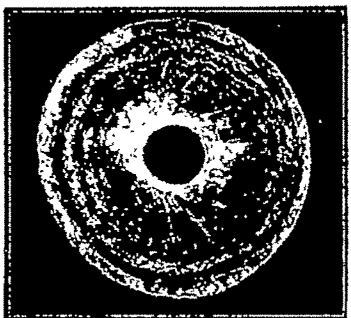


Fig. 1.

Fig 2.

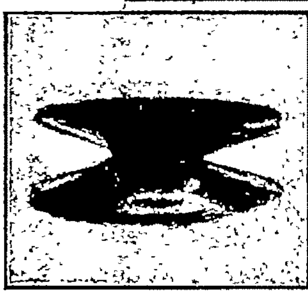


Fig 3.

Prof. Raman's photographs of percussion figures in Isotropic Elastic Solids.

13

Percussion Figures in Isotropic Elastic Solids.

BY

Sudhansukumar Banerji.

[*Read August 14th, 1921.*]

In a note published in *Nature* (October 9, 1919), Prof. C. V. Raman has given some photographs illustrating the manner in which rupture takes place in an isotropic elastic solid under the stresses set up by impact when these exceed the limits of perfect recovery. His pictures of the percussion figure produced on the surface of a thick glass plate by the impact of a polished hard steel ball are reproduced in Figs. 1, 2 and 3 given in the plate. "The fracture occurs," writes Prof. Raman, "at or near the margin of the compressed area in the form of a fine circular crack which spreads inwards into the plate obliquely in the form of a surface of revolution. This is clearly shown in Fig. 1 which is a front view of the percussion figure in reflected light, the dark circle in the middle being the uninjured area of contact between the ball and the plate. Fig. 2 is a side view of the internal fracture seen through the edge of the plate, the lower half of the picture being the image of the upper half formed by the reflection of light at the interior surface of the plate." Prof. Raman concludes his note with the remark that it seems clear that the internal fracture practically occurs along the surface of maximum shearing stress set up during the impact. It appears that this remark agrees with the suggestion made by Coulomb[1] long ago that the greatest shear produced in the material is a measure of the tendency to rupture. It is the object of the present note to point out that the observations made by Prof. Raman can be explained on a hypothesis concerning the conditions of rupture not very different from the one suggested by him. This hypothesis first enunciated by Tresca followed by G. H. Darwin[2] makes the maximum difference of the greatest and least principal stresses the measure of tendency to rupture.

[1] Coulomb, "Essai sur une application des regles de Maximis etc.," *Mem. par divers savans*, 1776, see also Love's *Elasticity* (third editon), pp. 119-20.

[2] Darwin, "On the stresses produced in the interior of the earth by the weight of continents and mountains," *Phil. Trans. Roy. Soc.*, Vol. 173 (1882). The same measure is adopted in the account of Prof. Darwin's work in Kelvin and Tait's *Nat. Phil.*, Part V, art. 832.

And as is well known the shear of any two rectangular lines intersecting at any point is equal to the difference between the elongations along the internal and external bisectors of the angle between them. As a matter of fact, from experiments on metal tubes subjected to various systems of combined stress J. J. Guest has concluded that the stress difference hypothesis is the one which accords best with observed results.[1]

The nature of the fracture cannot, however, be easily explained from Hertz's[2] drawing of the lines of principal stress in the plane passthrough the line of impact, for the reason that his drawing was in part conjectural and misleading, as the differential equation determining the direction of the lines of of principal stress cannot be integrated exactly. His drawing is reproduced in Fig. 4. His results mainly are that near

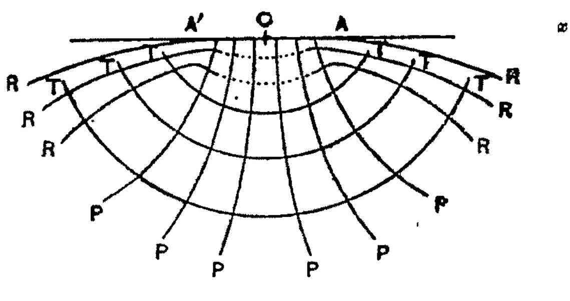


Fig. 4

the centre of the compressed area the principal planes of stress are nearly parallel to the coordinate planes. As we go from the centre of the compressed area along the axis of x, the component traction that is nearly parallel to the surface falls to zero, changes to tension and increases to a maximum near the edge of the compressed area ; it then diminishes more gradually without changing sign again. The other component is pressure, which continually diminishes as we go into the interior of the body along a line of stress starting near the centre of the compressed area. In Fig. 4, 0 is the centre of the compressed area, AA′ is the trace of this area on the plane of (x, z) ; lines like

[1] *Phil. Mag.* (Ser 5), Vol. 48 (1900). Mohr has criticised Guest (Zeitschr. des Vereines Deutschen Ingamenure, Bd. 44 (1900).

[2] See Love's Elasticity (second edition), p. 195 and also Hertz, *Ges. Werke*, Bd. 1, p. 174.

those ending at P are lines of pressure throughout, lines like those ending at T are lines of tension throughout, the lines ending at R are lines of stress in which the traction in the central (dotted) portion is pressure, and in the remaining portions is tension.

A more exact result has, however, been obtained by S. Fuchs[1] by a method of approximate integration in the case of a sphere resting on a plane. The diagram given by him differs in many essential respects from that of Hertz. The lines of principal stress in the body are represented in Fig. 5, where the full curved lines are the lines of principal stress along which the traction is pressure, and the dotted lines are lines of principal stress along which the traction is tension. It will be observed that near the compressed area both the principal stresses are pressures. This feature is also apparent from Hertz's diagram. This explains as noted by Prof. Raman that the fracture originates at or near the margin of the compressed area. A little further away one set of lines shows tension near the surface and pressure in the central portions. Still further away the same set of lines shows tension throughout.

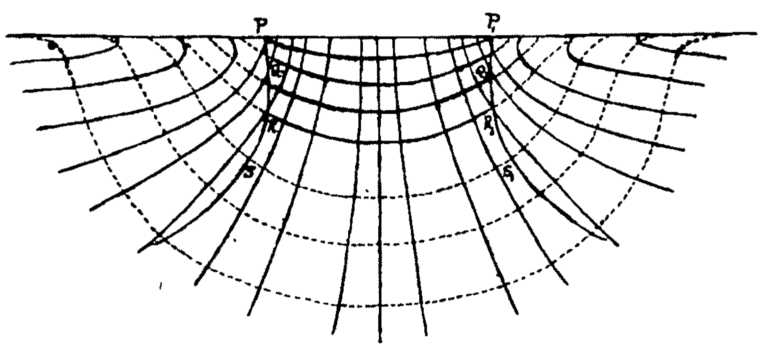


Fig. 5.

Since at the points P, Q, R, etc. and P_1, Q_1, R_1, etc, noted in the diagram (Fig. 5), one of the principal stresses changes from pressure to tension, this principal stress must be zero at these points. Consequently the stress difference must be maximum at or very near these points. Rupture will therefore take place along a line passing through these points. If therefore a surface be supposed to be generated by

[1] Fuchs, *Physikalische Zeitschr.*, 1913, p. 1282 See Love's *Elasticity*, (third edition), p. 196.

(*i*) On the Evaluation of some Recurrents and Bigradients (*ii*) and on the Expansion of some Functions as Power-series.

By

Haripada Datta, M.A. (Edin.)

[*Read, March 13th, 1921*].

I. Let the recurrent

$$\begin{vmatrix} 1 & 1 & 0 & 0 & 0 & \cdots & 0 \\ b_1 & a_1 & 1 & 0 & 0 & \cdots & 0 \\ b_2 & a_2 & a_2 & 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ b_n & a_n & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}$$

be denoted by β_{n+1} and the bigradient

$$\begin{vmatrix} 1 & 0 & 0 & 0 & 1 \\ b_1 & 1 & 0 & 1 & a_1 \\ b_2 & b_1 & 1 & a_1 & a_2 \\ b_3 & b_2 & b_1 & a_1 & a_3 \\ b_4 & b_3 & b_2 & a_3 & a_4 \end{vmatrix}$$

by $|\, (b_4)_3 \quad (a_2)_2 \,|$,

where 3 denotes the number of b—columns and 2 denotes the number of a—columns.

Now it is to be shown that the per-symmetric determinant of order n, *viz.*

$$\begin{vmatrix} \beta_1 & \beta_2 & \cdots & \beta_n \\ \beta_2 & \beta_3 & \cdots & \beta_{n+1} \\ \cdots & \cdots & \cdots & \cdots \\ \beta_n & \beta_{n+1} & \cdots & \beta_{2n-1} \end{vmatrix}$$

$$=(-1)^{(n-1)} \,|\, (b_{2n-2})_n \quad (a_{2n-2})_{n-1} \,|$$

i.e., a bigradient of order $(2n-1)$.

Let us take a determinant of order 4, *viz.*

$$\begin{vmatrix} \beta_1 & \beta_2 & \beta_3 & \beta_4 \\ \beta_2 & \beta_3 & \beta_4 & \beta_5 \\ \beta_3 & \beta_4 & \beta_5 & \beta_6 \\ \beta_4 & \beta_5 & \beta_6 & \beta_7 \end{vmatrix}$$

Performing on this determinant the operations

$$\begin{aligned} \text{row}_4 - a_1\,\text{row}_3 + a_2\,\text{row}_2 - a_3\,\text{row}_1 &= -\text{row}'_4 \\ \text{row}_3 - a_1\,\text{row}_2 + a_2\,\text{row}_1 &= \text{row}'_3 \\ \text{row}_2 - a_1\,\text{row}_1 &= -\text{row}'_2 \end{aligned}$$

we obtain

$$(-1)^2 \begin{vmatrix} \text{row}_1 \\ \text{row}'_2 \\ \text{row}'_3 \\ \text{row}'_4 \end{vmatrix} = (-1)^2 \text{D}.$$

The determinant D, when bordered with 3 columns and 3 rows, becomes

$$\begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ b_1 & 1 & 0 & 0 & 0 & 0 & 0 \\ b_2 & b_1 & 1 & 0 & 0 & 0 & 0 \\ b_3 & b_2 & b_1 & & & & \\ b_4 & b_3 & b_2 & & \text{D} & & \\ b_5 & b_4 & b_3 & & & & \\ b_6 & b_5 & b_4 & & & & \end{vmatrix}$$

Now by

$$\begin{aligned} &(i)\ \ \text{col}_5 + \text{col}_3 = \text{col}'_5 \\ &(ii)\ \ \text{col}_6 - \beta_2\,\text{col}'_5 = -\text{col}'_6 \\ &\qquad \text{col}'_6 + \text{col}_2 = \text{col}''_6 \\ &(iii)\ \ \text{col}^7 - \beta_3\,\text{col}'_6 = -\text{col}'_7 \\ &\qquad \text{col}'_7 - \beta_2\,\text{col}'_6{}' = -\text{col}'_7{}' \\ &\qquad \text{col}'_7{}' + \text{col}_1 = \text{cel}'_7{}'' \end{aligned}$$

we obtain $-|(b_4)_4(a_4)_3|$

If the persymmetric determinant is of order n, then the bigradient is of order $(2n-1)$.

The number of additional columns is $(n-1)$.

The first set of operations is

$$\text{row}_n - a_1\text{row}_{n-1} + a_2\text{row}_{n-1} - \ \dots\ + (-1)^{n-1}a_{n-1}\text{row}_1 = (-1)\text{row}_n'^{\,n-1}$$

$$\text{row}_{n-1} - a_1\text{row}_{n-1} + \ \dots\ + (-1)^{n-2}a_{n-2}\text{row}_1 = (+1)^{n-2}\text{raw}_{n-1}'$$

$$\dots \qquad \dots \qquad \dots \qquad \dots \qquad \dots \qquad \dots \qquad \dots$$

$$\text{row}_2 - a_1\text{row}_1 = -\text{row}_2'$$

And the second sets of operations are

$$(1)\quad \text{col}_{n+1} + \text{col}_{n-1} = \text{col}_{n+1}'$$

$$(2)\quad \text{col}_{n+2} - \beta_2\text{col}_{n+1}' = -\text{col}_{n+2}'$$
$$\text{col}_{n+2}' + \text{col}_{n+2} = \text{cel}_{n+2}''$$

$$(3)\quad \text{col}_{n+3} - \beta_3\text{col}_{n+1}' = -\text{col}_{n+3}'$$
$$\text{col}_{n=3}' - \beta_2\text{col}_{n+2}'' = -\text{col}_{n+3}''$$
$$\text{col}_{n+3}'' + \text{col}_{n-3} = \text{col}_{n+3}'''$$

$$(4)\quad \text{col}_{n+4} - \beta_4\text{col}_{n+1}' = -\text{col}_{n+4}'$$
$$\text{col}_{n+4}' - \beta_3\text{col}_{n+2}'' = -\text{col}_{n+4}''$$
$$\text{col}_{n+4}'' - \beta_2\text{col}_{n+3}''' = -\text{col}_{n+4}'''$$
$$\text{col}_{n+4}''' + \text{col}_{n-4} = -\text{col}_{n+4}^{v}$$

and so on.

The persymmetric determinant

$$\begin{vmatrix} \beta_r & \beta_{r+1} & \dots & \beta_{r+n-1} \\ \beta_{r+1} & \beta_{r=2} & \dots & \beta_{r+n} \\ \dots & \dots & \dots & \dots \\ \beta_{r+n-1} & \beta_{r+n} & \dots & \beta_{n+r-2} \end{vmatrix}$$

$$=(-1)^{n(r-1)}\left|\ (b_{2n+r-3})_n(a_{2n-r-2})_{n+r-2}\ \right|_{2n+r-2}$$

Even if the persymmetry of the determinant is broken by deleting any column and any row, still the determinant thus obtained, can also be transformed into a bigradient.

As for example

$$\begin{vmatrix} 1 & \beta_2 & \beta_5 \\ \beta_2 & \beta_3 & \beta_6 \\ \beta_4 & \beta_5 & \beta_8 \end{vmatrix} = \begin{vmatrix} 0 & 1 & \beta_2 & \beta_5 \\ 0 & \beta_2 & \beta_3 & \beta_6 \\ 1 & \beta_3 & \beta_4 & \beta_7 \\ 0 & \beta_4 & \beta_5 & \beta_1 \end{vmatrix}$$

$$= -\begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ b_1 & 0 & 0 & 0 & 0 & 0 & 1 & a_1 \\ b_2 & 0 & 0 & 0 & 0 & 1 & a_1 & a_2 \\ b_3 & 1 & 0 & 0 & 1 & a_1 & a_2 & a_3 \\ b_4 & b_1 & 0 & 1 & a_1 & a_3 & a_3 & a_4 \\ b_5 & b_3' & 0 & b_1 & a_2 & a_3 & a_4 & a_5 \\ b_6 & b_3 & 1 & b_2 & a_3 & a_4 & a_5 & a_6 \\ b_7 & b_4 & a_1 & b_3 & a_4 & a_5 & a_6 & a_7 \end{vmatrix}$$

If the persymmetry of the determinant is broken by deleting any number of columns and the same number of consecutive rows from the bottom (or from the top), the determinant thus obtained, can be transformed into a bigradient.

II. Now we consider an example of the forgoing theorem.

Let $b_n = y^n/n!$ and $a_n = x^n/n!$

It is evident that the recurrent β_6

$$= \frac{1}{5!\ 4!\ 3!\ 2!}$$

multiplied by the determinant

$$\begin{vmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ y & x & 1 & 0 & 0 & 0 \\ y^2 & x^2 & 2x & 2 & 0 & 0 \\ y^3 & x^3 & 3x^2 & 6x & 6 & 0 \\ y^4 & x^4 & 4x^3 & 12x^2 & 24x & 24 \\ y^5 & x^5 & 5x^4 & 20x^3 & 60x^2 & 120x \end{vmatrix}.$$

This determinant $=(x-y)$

$$\times\begin{vmatrix} 1 & 1 & 0 & 0 & 0 \\ x+y & 2x & 2 & 0 & 0 \\ x^2+xy+y^2 & 3x^2 & 6x & 6 & 0 \\ x^3+x^2y+xy^2+y^3 & 4x^3 & 12x^2 & 24x & 25 \\ x^4+x^3y+x^2y^2+xy^3+y^4 & 5x^4 & 20x^3 & 60x^2 & 120x \end{vmatrix}$$

By the operations

$$\text{row}_5-x\,\text{row}_4,\ \text{row}_4-x\,\text{row}_3,\ \text{row}_3-x\,\text{row}_2,\ \text{row}_2-x\,\text{row}_1$$

the last determinant becomes

$$\begin{vmatrix} 1 & 1 & 0 & 0 & 0 \\ y & x & 2 & 0 & 0 \\ y^2 & x^2 & 4x & 6 & 0 \\ y^3 & x^3 & 6x^2 & 18x & 24 \\ y^4 & x^4 & 8x^3 & 36x^2 & 96x \end{vmatrix}$$

$$=2.3.4\begin{vmatrix} 1 & 1 & 0 & 0 & 0 \\ y & x & 1 & 0 & 0 \\ y^2 & x^2 & 2x & 2 & 0 \\ y^3 & x^3 & 3x^2 & 6x & 6 \\ y^4 & x^4 & 4x^3 & 12x^2 & 24x \end{vmatrix}$$

$=4!\ 2!\ 3!\ 4!\ \beta_5$

$=$Hence $\beta_6=\frac{x-y}{5}\beta_5=\frac{1}{5!}(x-y)^5$

Similarly $\beta_{n+1}=\frac{1}{n!}(x-y)^n$

Therefore the bigradient

$$\begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ y & 1 & 0 & 0 & 0 & 1 & x \\ \frac{y^2}{2!} & y & 1 & 0 & 1 & x & \frac{x^2}{2!} \\ \frac{y^3}{3!} & \frac{y^2}{2!} & y & 1 & x & \frac{x^2}{2!} & \frac{x^3}{3!} \\ \frac{y^4}{4!} & \frac{y^3}{3!} & \frac{y^2}{2!} & y & \frac{x^2}{2!} & \frac{x^3}{3!} & \frac{x^4}{4!} \\ \frac{y^5}{5!} & \frac{y^4}{4!} & \frac{y^3}{3!} & \frac{y^2}{2!} & \frac{x^3}{3!} & \frac{x^4}{4!} & \frac{x^5}{5!} \end{vmatrix}$$

$$=\begin{vmatrix} 1 & x-y & \frac{(x-y)^2}{2!} & \frac{(x-y)^3}{3!} \\ x-y & \frac{(x-y)^2}{2!} & \frac{(x-y)^3}{3!} & \frac{(x-y)^4}{4!} \\ \frac{(x-y)^2}{2!} & \frac{(x-y)^3}{3!} & \frac{(x-y)^4}{4!} & \frac{(x-y)^5}{5!} \\ \frac{(x-y)^3}{3!} & \frac{(x-y)^4}{4!} & \frac{(x-y)^5}{5!} & \frac{(x-y)^6}{6!} \end{vmatrix}.$$

This determinant has already been evaluated.*

and it$=\dfrac{1}{1.2.^2 3.^3 4.^4 5.^3 6.^2 7}(x-y)^{12}$.

Similarly bigradients of higher order can be evaluated

(*ii*) Let $b_n=\dfrac{1}{(a-1)(a^2-1)\ldots(a^n-1)}\,y^n$

and $a_n=\dfrac{1}{(a-1)(a^2-1)\ldots(a^n-1)}\,x^n$

* *On the Theory of continued Fractions*, see (c) *Proc. Edin. Math. Soc. Part* (*II*), (1915-16).

Here $\beta_4 = \dfrac{1}{(a-1)^3(a^2-1)^2(a^3-1)}$

$$\times \begin{vmatrix} 1 & 1 & 0 & 0 \\ y & x & a-1 & 0 \\ y^2 & x^2 & (a^2-1)x & (a-1)(a^2-1) \\ y^3 & x^3 & (a^3-1)x^2 & (a^2-1)(a^3-1)x \end{vmatrix}$$

This recurrent $=(x-y)$

$$= \begin{vmatrix} 1 & a-1 & 0 \\ x+y & (a^2-1)x & (a-1)(a^2-1) \\ x^2+xy+y^2 & (a^3-1)x^2 & (a^2-1)(a^3-1)x \end{vmatrix}$$

This determinant again

$$= \frac{1}{a \cdot a^2} \begin{vmatrix} 1 & a-1 & 0 \\ z+ay & (a^2-1)z & (a-1)(a^2-1)a \\ z^2+ayz+y^2 & (a^3-1)z & (a^2-1)(a^3-1)az \end{vmatrix}$$

where $z = ax$.

The last determinant transforms into

$$\begin{vmatrix} 1 & a-1 & 0 \\ y & (a-1)z & (a-1)(a^2-1) \\ y^2 & (a-1)z^2 & (a^2-1)(a^2-1)z \end{vmatrix}$$

which $=(a-1)(a^2-1) \begin{vmatrix} 1 & 1 & 0 \\ y & z & a-1 \\ y^2 & z^2 & (a^2-1)z \end{vmatrix}$

Thus we see that

$$\beta_4 = \frac{x-y}{a^3-1} \cdot \frac{z-y}{a^2-1} \cdot \frac{w-y}{a-1}$$

$$= \frac{(x-y)(ax-y)(a^2x-y)}{(a-1(a^2-1)(a^3-1)}$$

Subtracting the last column from each of the other columns of the above diterminant, we have this determinant

$$= -\frac{(a-1)(a^2-1)(a^3-1)}{(a^4-1)(a^5-1)(a^6-1)}\, a^9 \times \begin{vmatrix} \dfrac{1}{a-1} & \dfrac{1}{a^2-1} & \dfrac{1}{a^3-1} \\[2ex] \dfrac{1}{a^2-1} & \dfrac{1}{a^3-1} & \dfrac{1}{a^4-1} \\[2ex] \dfrac{1}{a^3-1} & \dfrac{1}{a^4-1} & \dfrac{1}{a^5-1} \end{vmatrix}$$

Performing on this determinant the operations

$$\mathrm{col}_1 - a^2\,\mathrm{col}_3 = (a^2-1)\,\mathrm{col}'_1$$

$$\mathrm{col}_2 - a\,\mathrm{col}_3 = (a-1)\,\mathrm{col}'_2$$

we obtain

$$\frac{(a^2-1)(a-1)}{(a^3-1)(a^4-1)(a^5-1)} \begin{vmatrix} \dfrac{1}{a-8} & \dfrac{1}{a^2-1} & 1 \\[2ex] \dfrac{1}{a^2-1} & \dfrac{1}{a^3-1} & 1 \\[2ex] \dfrac{1}{a^3-1} & \dfrac{1}{a^4-1} & 1 \end{vmatrix}$$

This determinant* again

$$= \frac{(a^2-1)(a-1)}{(a^3-1)(a^4-1)}\, a^4 \begin{vmatrix} \dfrac{1}{a-1} & \dfrac{1}{a^2-1} \\[2ex] \dfrac{1}{a^2-1} & \dfrac{1}{a^3-2} \end{vmatrix}$$

* *See the paper "on the Theory of continued Fractions, Proc. Edin. Math. Soc. Part (II) (1915-16.)*

the last determinant becomes

$$=\left(\frac{x-ay}{a}\right)^3\left(\frac{x-a^2y}{a^2}\right)^2\frac{x-a^3y}{a^3}$$

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ \frac{1}{a-1} & \frac{1}{a^2-1} & \frac{1}{a^3-1} & \frac{1}{a^4-1} \\ \frac{1}{(a-1)(a^2-1)} & \frac{1}{(a^2-1)(a^3-1)} & \frac{1}{(a^3-1)(a^4-1)} & \frac{1}{(a^4-1)(a^5-1)} \\ \frac{1}{(a-1)\dots(a^3-1)} & \frac{1}{(a^2-1)\dots(a^4-1)} & \frac{1}{(a^3-1)\dots(a^5-1)} & \frac{1}{(a^4-1)\dots(a^5-1)} \end{vmatrix}.$$

Now by

(1) $(1+a^2)\text{row}_4+\text{row}_3=a^2\text{row}_4'$

$(1-a)\text{row}_3+\text{row}_2=a\text{row}_3'$

(1) $(1-a)\text{row}_4'+\text{row}_3'=a\text{row}_4''$

the last determinant reduces to

$$\frac{a^2}{1-a^2}\cdot\frac{a}{1-a}\cdot\frac{a}{1-a}$$

$$\times\begin{vmatrix} 1 & 1 & 1 & 1 \\ \frac{1}{a-1} & \frac{1}{a^2-1} & \frac{1}{a^3-1} & \frac{1}{a^4-1} \\ \frac{1}{a^2-1} & \frac{1}{a^3-1} & \frac{1}{a^4-1} & \frac{1}{a^5-1} \\ \frac{1}{a^3-1} & \frac{1}{a^4-1} & \frac{1}{a^5-1} & \frac{1}{a^6-1} \end{vmatrix}.$$

Similarly $\beta_{n+1}=\frac{1}{a^n-1}\beta_n$

$$=\frac{(x-y)(ax-y)\dots(a^{n-1}x-y)}{(a-1)(a^2-1)\dots(a^n-1)}. \quad \dots \quad (1)$$

Hence the bigradient of order 7 consisting of $4b-$ columns and $3a-$ columns, becomes the persymmetric determinant which

$$=(x-y)^3(ax-y)^2(a^2x-y)\,\frac{1}{(a-1)^3(a^2-1)^2(a^3-1)}\times$$

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ \frac{x-y}{a-1} & \frac{an-y}{a^2-1} & \frac{a^2x-y}{a^3-1} & \frac{a^3x-y}{a^4-1} \\ \frac{(x-y)(ax-y)}{(a-1)(a^2-1)} & \frac{(ax-y)(a^2x-y)}{(a^2-)(a^3-1)} & \frac{(a^2x-y)(a^3n-y)}{(a^3-1)(a^4-1)} & \frac{(a^3x-y)(a^4n-y)}{(a^4-1)(a^5-1} \\ \frac{(x-y)\dots(a^2x-y)}{(a-1)\dots(a^3-1)} & \frac{(ax-y)\dots(a^3x-y)}{(a^2-1)\dots(a^4-1)} & \frac{(a^2x-y)\dots(a^4x-y)}{(a^3-1)\dots(a^5-1)} & \frac{(a^3x-y)\dots(a^5n-y)}{(a^4-1)\dots(a^6-1)} \end{vmatrix}$$

By the operations

(i) $a\,\mathrm{row}_1-x\,\mathrm{row}_3=(x-ay)\mathrm{row}'_4$

$a\,\mathrm{row}_3-x\,\mathrm{row}_2=(n-ay)\mathrm{row}'_3$

$a\,\mathrm{row}_2-x\,\mathrm{row}_1=(n-ay)\mathrm{row}'_2$

(ii) $a^2\text{row}_4' - x\text{row}_3' = (x-a^2y)\text{row}_4''$

$a^2\text{row}_3' - x\text{row}_2' = (x-a^2y)\text{row}_3''$

(iii) $a^3\text{row}_4'' - x\text{row}_3'' = (x-a^3y)\text{row}_4'''$

Hence the bigradient

$$=a^8\frac{[(x-y)^3(ax-y)^2(a^2-y)][(x-ay)^3(x-a^2y)^2(x-a^3y)]}{(a-1)(a^2-1)^2(a^3-1)^3(a^4-1)^3(a^5-1)^2(a^6-1)}$$

The bigradient of order $(2n-1)$

$$=\frac{[(x-y)^{n-1}(ax-y)^{n-2}\ldots(a^{n-2}x-y)][(x-ay)^{n-1}\ldots(x-a^{n-1}y)]}{(a-1)(a^2-1)^2\ldots(a^{n-1}-1)^{n-1}(a^n-1)^{n-1}(a^{n+1}-1)^{n-2}\ldots(a^{2n-2}-1)}$$

$$\times a^{\frac{1}{3}(n-2)(n-1)n}$$

III. From (1) we see that

$$\frac{1+\frac{1}{a-1}yz+\frac{1}{(a-1)(a^2-1)}y^2z^2+\quad\ldots}{1+\frac{1}{a-1}xz+\frac{1}{(a-1)(a^2-1)}x^2z^2+\quad\ldots}$$

$$=1+\frac{y-x}{a-1}z+\frac{(y-x)(y-ax)}{(a-1)(a^2-1)}z^2+\qquad\ldots\quad(3)$$

Therefore

$$\left(1+\frac{y-x}{a-1}z+\frac{(y-x)(y-ax)}{(a-1)(a^2-1)}z^2+\quad\ldots\right)^{-1}$$

$$=1+\frac{x-y}{a-1}z+\frac{(x-o)(x-ay)}{(a-1)(a^2-1)}z^2+\qquad\ldots\quad(4)$$

From (2) we have

$$\left\{1+\frac{1}{a-1}x+\frac{1}{(a-1)(a^2-1)}a^2+\quad\ldots\right\}^{-1}$$

$$=1-\frac{1}{a-1}x+\frac{1}{(a-1)(a^2-1)}ax^2\ldots$$

$$+(-1)^n\frac{1}{(a-1)\ldots(a^n-1)}a^{\frac{1}{2}n(n-1)}x^n+\quad\ldots$$

Let the series, on the left hand side, be denoted by $f(a, x)$ and the series on the right-hand side by $\phi(a, -x)$.

Thus—

$$\frac{f(a,x)}{f(a,y)}=\frac{\phi(a,-y)}{\phi(a,-x)} \qquad \ldots \quad (4)$$

Therefore

$$\beta_n=\begin{vmatrix} 1 & 1 & 0 & \ldots & 0 \\ \frac{y}{a-1} & \frac{x}{a-1} & 1 & \ldots & 0 \\ \frac{ay^2}{(a-1)(a^2-1)} & \frac{ax^2}{(a-1)(a^2-1)} & \frac{x}{a-1} & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix}_{n+1}$$

$$=\frac{(y-x)(ay-x)\ \ldots\ (a^{n-2}y-x)}{(a-1)(a^2-1)\ \ldots\ (a^n-1)} \qquad \ldots \quad (5)$$

To prove this independently of the other, we have

$$\beta_4=\frac{x-y}{(a-1)^3(a^2-1)^2(a^3-1)}$$

$$\times\begin{vmatrix} 1 & a-1 & 0 \\ an+z & x(a^2-1) & (a-1)(a^2-1) \\ a^3x^2+a^2xz+az^2 & ax(a^3-1) & a(a^2-1)(a^3-1) \end{vmatrix}$$

This determinant of order 3

$$=\begin{vmatrix} 1 & 1 & 0 \\ z & x & a-1 \\ az^2 & ax^2 & (a^2-1)x \end{vmatrix}$$

$$\therefore \beta_4=\frac{(x-y)(x-z)(x-w)}{(a^3-1)(a^2-1)(a-1)}$$

$$=\frac{(x-y)(x-ay)(x-a^2y)}{(a-1)(a^2-1)(a^3-1)}$$

And

$$\begin{vmatrix} \dfrac{x}{a-1} & 1 & 0 \\ \dfrac{x^2}{(a-1)(a^2-1)} & \dfrac{x}{a-1} & 1 \\ \dfrac{x^3}{(a-1)\ldots(a^3-1)} & \dfrac{x^2}{(a-1)(a^2-1)} & \dfrac{x}{a-1} \end{vmatrix}$$

$$= \frac{1}{(a-1)^3(a^2-1)^2(a^3-1)} \begin{vmatrix} x & a-1 & 0 \\ x^2 & (a^2-1)x & (a-1)(a^2-1) \\ x^3 & (a^3-1)x^2 & (a^2-1)(a^3-1)x \end{vmatrix}$$

This determinant

$$= \begin{vmatrix} x & a-1 & 0 \\ 0 & a(a-1)x & (a-1)(a^2-1) \\ 0 & a^2(a-1)x^2 & (a^2-1)(a^2-1)ax \end{vmatrix}$$

$$= a^2x(a-1)(a^2-1) \begin{vmatrix} x & a-1 \\ x^2 & (a^2-1)x \end{vmatrix}$$

$$\therefore \beta_3 = \frac{a^2x}{a^3-1} \cdot \frac{ax^2}{(a-1)(a^2-)}$$

Similarly

$$\beta_n = \frac{1}{(a-1)(a^2-1)\ldots(a^n-1)} a^{\frac{1}{2}n(n-1)} x^n \qquad \ldots \quad (6)$$

$$\text{IV.} \quad \frac{(1-xy)(1-axy)(1-a^2xz)\ldots}{(1-yz)(1-ayz)(1-a^2yz)\ldots} = \frac{\phi(a,xz)}{\phi(a,yz)}$$

$$= 1 - \frac{y-x}{a-1} z + \frac{(y-x)(y-ax)}{(a-1)(a^2-1)} z^2 - \qquad \ldots \quad (7)$$

$$\therefore \frac{(1-xz)(1-a^2xz)(1-a^4xz)\ldots}{(1-yz)(1-a^2yz)(1-a^4yz)\ldots}$$

$$= 1 - \frac{y-x}{a^2-1} z + \frac{(y-n)(y-a^2x)}{(a^2-1)(a^4-1)} z^2 - \frac{(y-n)(y-a^2x)(y-a^4n)}{(a^2-1)(a^4-1)(a^6-1)} + \ldots \quad (8)$$

$$\therefore \frac{(1-ax)(1-a^3x)(1-a^5x)\ldots}{(1-a)(1-a^3)(1-a^5)\ \ldots}$$

$$=1-\frac{1-1}{a^2-1}a+\frac{(1-x)(1-a^2x)}{(a^2-1)(a^4-1)}a^2-\frac{(1-x)(1-a^2x)(1-a^4x)}{(a^2-1)(a^4-1)(a^6-1)}+\ldots \quad (9)$$

Hence

$$\frac{(1-a^2)(1-a^4)(1-a^6)\ \ldots}{(1-a)(1-a^3)(1-a^5)\ \ldots}$$

$$=1+\frac{a-1}{a^2-1}a+\frac{(a-1)(a^3-1)}{(a^2-1)(a^4-1)}a^2+\frac{(a-1)(a^3-1)(a^5-1)}{(a^2-1)(a^4-1)(a^6-1)}a^3+\ldots \quad (10)$$

But also

$$\frac{(1-ax)(1-a^3x)(1-a^5x)\ \ldots}{(1-a)(1-a^3)(1-a^5)\ \ldots}\text{*}$$

$$=1+\frac{1-x}{1-a}a+\frac{(1-x)(1-ax)}{(1-a)(1-a^2)}a^3+\frac{(1-x)(1-ax)(1-a^2x)}{(1-a)(1-a^2)(1-a^3)}a^6+\ldots \quad (11)$$

$$\therefore \frac{(1-a^2)(1-a^4)(1-a^6)\ \ldots}{(1-a)(1-a^3)(1-a^5\ \ \ldots}$$

$$=1+a+a^3+a^6+\ \ldots\ +a^{\frac{1}{2}(n-1)n}+\ \ldots \quad (12)$$

Hence from (10) and (12), we obtain

$$1+\frac{a-1}{a^2-1}a+\frac{(a-1)(a^3-1)}{(a^2-1)(a^4-)1}a^2+=1+a+a^3+a^6+a^{10}+\ldots \quad (13)$$

From (9) and (11), equating the co-efficients of x, we see that

$$\frac{a}{a^2-1}\left\{1-\frac{1}{a^2-1}a+\frac{1}{(a^2-1)(a^4-1)}a^2-\ \ldots\right\}$$

$$\frac{a}{a-1}\left\{1-\frac{1}{a-1}a^2+\frac{1}{(a-1)(a^2-1)}a^5-\frac{1}{(a-1)(a^2-1)(a^3-1)}a^9+\ldots\right\}$$

$$\ldots \quad (14)$$

From (9) and (11)

$$\left\{1+\frac{1-x}{a^2-1}a+\frac{(1-x)(a^2-x)}{(a^2-x)(a^4-1)}a^2+\frac{(1-n)(a^2-n)(a^4-n)}{(a^2-1)(a^4-1)(a^6-1)}a^3+\ldots\right\}^{-1}$$

$$=1+\frac{1-x}{1-a}a+\frac{(1-x)(1-ax)}{(1-a)(1-a^2)}a^3+\frac{(1-n)(1-an)(1-a^2n)}{(1-a)(1-a^2)(1-a^3)}a^6+\ldots$$

* [*Jacobi :—Ges. Werke*, p. 237].

Hence

$$\{f(a,x)f(a,-x)f(a,ix)f(a,-ix)\}$$
$$\times\{f(a^2,x^2)f(a^2,-x^2)f(a^2,-ix^2)f(a^2,-ix^2)\}$$
$$=f(a^8,-x^8). \quad \dots \quad (56)$$

Similarly

$$\phi(a,x)\phi(a,x)$$
$$=1+\frac{1}{a^2-1}x^2+\frac{1}{(a^2-1)(a^4-1)}a^2x^4+\dots$$
$$=\phi(a^2,x^2) \quad \dots \quad (17)$$

and

$$\phi(a,x)\phi(a,-x)\phi(a,ix)\phi(a,-ix)$$
$$=\phi(a^4,x^4). \quad \dots \quad (18)$$

As

$$\frac{1}{(1-x)(1-an)(1-a^2n)\dots}$$
$$=1-\frac{1}{a-1}x+\frac{1}{(a-1)(a^2-1)}x^2 \quad \dots \quad (19)$$

Therefore

$$\frac{1}{(1-a)(1-a^2)(2-a^2)\dots}$$
$$=1-\frac{1}{a-1}a+\frac{1}{(a-1)(a^2-1)}. \quad \dots \quad (20)$$

From (17)

$$\frac{1}{(a^{k+2}-1)(a^{n+2}-1)\dots(a^{2n}-1)}a^{n^2}$$
$$-\frac{1}{(a-1)\{(a^{n+1}-1)\dots(a^{2n-1}-1)\}}a^{(n-1)^2}+\dots+(-1)^{n-1}$$
$$\frac{1}{(a-1)\dots(a^{n-1}-1)\{(a^{n+1}-1\}}a+(-1)^n\frac{1}{2(a-1)(a^2-1)\dots(a^n-1)}$$
$$=\frac{2}{2(a+1)(a^2+1)\dots(a^n+1)} \quad \dots \quad (21)$$

And from

$$\frac{1}{(a^{n+1}-1)\ldots(a^{2n}-1)}-\frac{1}{(a-1)\{(a^{n+1}-1)\ldots(a^{2n-1}-1)\}}$$

$$+\frac{1}{(a-1)(a^2-1)\{a^{n+1}-1)\ldots(a^{2n-2}-1)\}}-\ldots+(-1)^{n-1}$$

$$\frac{1}{(a-1)\ldots(a^{n-1}-1)\{(a^{n+1}-1)\}}+(-1)^n\frac{1}{2(a-1)\ldots(a^n-1)}$$

$$=\frac{1}{2(a+1)(a^2+1)\ldots(a+1^n)} \qquad \ldots \quad (22)$$

$$\left.\begin{array}{l}\text{When } n \text{ is even } (21)=(22)\\ \text{and when } n \text{ is odd } (21)=-(22)\end{array}\right\}$$

V. Let

$$\beta_4=\begin{vmatrix} 1 & 1 & 0 & 0\\ b_1 & 0 & 1 & 0\\ b_2 & a_2 & 0 & 1\\ b_3 & 0 & a_2 & 0\end{vmatrix}$$

$$=-\begin{vmatrix} 1 & 0 & 1 & 0\\ b_1 & 1 & 0 & 0\\ b_2 & 0 & a_2 & 1\\ b_3 & a_2 & 0 & 0\end{vmatrix}$$

$$=\begin{vmatrix} b_1 & 1 & 0 & 0\\ b_3 & a_2 & 0 & 0\\ 1 & 0 & 1 & 0\\ b_2 & 0 & a_2 & 1\end{vmatrix}=\begin{vmatrix} b_1 & 1\\ b_3 & a_2\end{vmatrix}$$

Similarly

$$\beta_{2n}=(-1)^n\begin{vmatrix} b_1 & 1 & 0 & 0 & 0 & \ldots\\ b_3 & a_2 & 1 & 0 & 0 & \ldots\\ b_5 & a_4 & a_2 & 1 & 0 & \ldots\\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots\end{vmatrix}_n=(-1)^n D_n$$

and

$$\beta_{2r-1}=(-1)^r\begin{vmatrix} 1 & 1 & 0 & 0 & \dots \\ b_2 & a_2 & 1 & 0 & \dots \\ b_4 & a_4 & a_2 & 1 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix}_r=(-1)^r B_n$$

If

$$b_n=\frac{1}{(a-1)(a^2-1)\dots(a^n-1)}$$

and

$$a_{2n}=\frac{1}{(1-a^2)(1-a^4)\dots(1-a^{2n})}$$

then

$$B_n=(-1)^r\frac{1}{(a-1)(a^2-1)\dots(a^{2r-2}-1)}a^{\frac{1}{2}(2r-3)(2r-2)}$$

and

$$D_n=(-1)^n\frac{1}{(a-1)(a^2-1)\dots(a^{2n-1}-1)}a^{\frac{1}{2}(2n-2)(2n-1)}$$

Similar results can also be obtained from (17).

VI. If

$$\beta_3=\begin{vmatrix} b_1 & 1 & 0 \\ b_2 & b_1 & 1 \\ b_3 & b_2 & b_1 \end{vmatrix}$$

then

$$\begin{vmatrix} b_1 & 1 & 0 & \dots & 0 \\ \beta_2 & b_1 & 1 & \dots & 0 \\ \beta_3 & \beta_2 & b_1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix}_n=(-1)^n b_n$$

For by

$$\text{row}_n-b_1\text{row}_{n-1}+b_2\text{row}_{n-2}-\dots+(-1)^n b_{n-1}\text{row}_1$$

we have all the elements of the last row. except the first one, zeros, and this element is $-b_n$

Ex. $$\begin{vmatrix} \frac{x-y}{a-1} & 1 & 0 & \ldots \\ \frac{(x-y)(ax-y)}{(a-1)(a^2-1)} & \frac{x-y}{a-1} & 1 & \ldots \\ \frac{(x-y)(ax-y)(a^2x-y)}{(a-1)(a^2-1)(a^3-1)} & \frac{(x-y)(ax-y)}{(a-1)(a^2-1)} & \frac{x-y}{a-1} & \ldots \\ \ldots & \ldots & \ldots & \end{vmatrix}$$

By

$$\text{row}_n+\frac{y-x}{a-1}\text{row}_{n-1}+\frac{(y-x)(ay-x)}{(a-1)(a^2-1)}\text{row}_{n-2}+\ldots$$

$$+\frac{(y-x)\ldots(a^{n-2}y-x)}{(a-1)\ldots(a^{n-1}-1)}\text{row}_1$$

we obtain the above recurrent

$$=(-1)^n\frac{(y-x)(ay-x)\ldots(a^{n-1}y-x)}{(a-1)(a^2-1)\ldots(a^n-1)}$$

for

$$\frac{(x-y)(ax-y)\ldots(a^{n-1}x-y)}{(a-1)(a^2-1)\ldots(a^n-1)}+\frac{y-x}{a-1}\cdot\frac{(x-y)(ax-y)\ldots(a^{n-2}x-y)}{(a-1)(a^2-1)\ldots(a^{n-1}+1)}\cdot$$

$$+\frac{(y-x)(ay-x)}{(a-1)(a^2-1)}\cdot\frac{(x-y)(ax-y)\ldots(a^{n-3}x-y)}{(a-1)(a^2-1)\ldots(a^{n-2}-y}+\ldots$$

$$+\frac{(y-x)(ay-x)\ldots(a^{n-2}y-x)}{(a-1)(a^2-)1\ldots(a^{n-1}-1)}\cdot\frac{x-y}{a-1}$$

$$=-\frac{(y-x)(ay-x)\ldots(a^{n-2}y-x)}{(a-1)(a^2-1)\ldots(a^n-1)}.$$

The above relation can be written as

$$T_n+S_1T_{n-1}+S_2T_{n-2}+\ldots+S_{n-1}T_1+S_n=0,$$

VII. Let

$$C_3^{1'}=\begin{vmatrix} 1 & 1 & 1 \\ \frac{1}{(1-a)(1-a^2)} & \frac{1}{1-a^2} & 1 \\ \frac{1}{(1-a)\ldots(1-a^4)} & \frac{1}{(1-a^3)(1-a^4)} & \frac{1}{1-a^2} \end{vmatrix}$$

$$C_3=\frac{1}{(1-a)(1-a^2)}\cdot\frac{1}{(1-a)(1-a^2)(1-a^3)(1-a^4)}$$

$$\times\begin{vmatrix} 1 & 1 & 0 \\ 1 & 1-a & (1-a)(1-a^3) \\ 1 & (1-a)(1-a^3 & (1-a)(1-a^3)(1-a^4) \end{vmatrix}.$$

Performing on this determinant the operations

$$\text{row}_3-\text{row}_2,\quad \text{row}_2-\text{row}_1$$

and then multiplying the third column by $(-a)$, we obtain this determinant

$$=-\frac{1}{a}\begin{vmatrix} -a & -a(1-a)(1-a^2 \\ -a^3(1-a) & -a^3(1-a)(1-a-a^2+a^5) \end{vmatrix}$$

$$=(-1)a^3\begin{vmatrix} 1 & 1-a^2 \\ 1-a & 1-a-a^2+a^5 \end{vmatrix}\times(1-a)$$

Again by $\text{row}_2-\text{row}_1$ this last determinant

$$=\begin{vmatrix} 1 & 1-a^2 \\ -a & -a(1-a^4) \end{vmatrix}$$

$$=-a(1-a^2)^2(1-a^4)\begin{vmatrix} \dfrac{1}{1-a^2} & 1 \\ \dfrac{1}{(1-a^2)(1-a^4)} & \dfrac{1}{1-a^2} \end{vmatrix}$$

$$=-a(1-a^2)^2(1-a^4)\frac{1}{(0-a^2)(1-a^4)}a^2\,[\text{by (6)}].$$

Hence

$$C_3=\frac{1}{(1-a)(1-a^2)\dots(1-a^4)}a^6$$

After multiplying by the denominators and then performing the operations

$\text{row}_4-\text{row}_3$, $\text{row}_4-\text{row}_2$, $\text{row}_2-\text{row}_1$, and taking $(1-a^4)(1-a)^2$ $(1-a^2)$ outside, we obtain from C_4, the determinant

$$\begin{vmatrix} -a & 1-a^2 & 0 \\ -a^3(1-a) & a^2(1-a-a^2+a^5) & 1-a^2 \\ -a^5(1-a)(1-a^3) & a^3(1-a^3)(1-a-a^2+a^7) & a^2(1-a^3-a^3+a^2) \end{vmatrix}.$$

Now multiply the second column by a and the third by a^3, then taking out the common factors, we have

$$\begin{vmatrix} 1 & 1-a^2 & 0 \\ 1-a & 1-a-a^2+a^3 & 1-a^2 \\ (1-a)(1-a^2) & (1-a^3)(1-a-a^2+a^7) & 1-a^3-a^4+a^0) \end{vmatrix}.$$

By $\text{row}_3-\text{row}_2$, $\text{row}_2-\text{row}_1$, then multiplying the third column by $(-a)$, we have

$$\begin{vmatrix} 1 & 1 & 0 \\ -a & -a(1-a^4) & -a(1-a^2) \\ -a^3(1-a) & -a^3(1-a-a^4+a^7) & -a^3(1-a-a^2+a^7) \end{vmatrix}.$$

Taking out $(-a)$, $(-a^3)$ and then by $\text{row}_3-\text{row}_2$, we have

$$\begin{vmatrix} 1 & 1-a^2 & 0 \\ 1 & 1-a^4 & 1-a^2 \\ -a & -a(1-a^6) & -a(1-a^6) \end{vmatrix}$$

$$=(-1)^4 a(a^2-1)^3(a^4-1)^2(a^6-1)\frac{1}{1-a^4}$$

$$\times\begin{vmatrix} \frac{1}{a^2-1} & 1 & 0 \\ \frac{1}{(a^2-1)(a^4-1)} & \frac{1}{a^2-1} & 1 \\ \frac{1}{(a^2-1)\ldots(a^6-1)} & \frac{1}{(a^2-1)(a^4-1)} & \frac{1}{a^2-1} \end{vmatrix}$$

This last determinant [by (6)]

$$=\frac{1}{(a^2-1)(a^4-1)(a^6-1)}\;a^6$$

Hence

$$C_4=-\frac{1}{(1-a)(1-a^2)\ldots(1-a^6)}\;a^{15}$$

Similarly

$$C_n=(-1)^{n-1}\frac{1}{(1-a)(1-a^2)\ldots(1-a^{2(n-1)})}a(n-1)(2n-3)\ldots \qquad (23)$$

Thus we obtain

$$\frac{1+\dfrac{1}{(1-a)(1-a^2)}\,x+\dfrac{1}{(1-a)(1-a^2)\dots(1-a^4)}\,x^2+}{1+\dfrac{1}{1-a^2}\,x+\dfrac{1}{(1-a^2)(1-a^4)}\,x^2+\dfrac{1}{(1-a^2)(1-a^4)(1-a^6)}\,x^3+\dots}$$

$$=1+\frac{1}{(1-a)(1-a^2)}\,ax+\frac{1}{(1-a)(1-a^2)(1-a^3)(1-a^4)}\,a^6x^2$$

$$+\frac{1}{(1-a)(1-a^2)\dots(1-a^6)}\,a^{15}x^2+\dots$$

$$+\frac{1}{(1-a)(1-a^2)\dots(1-a^{2n})}\,a^{n(2n-1)}x^n+\dots \qquad (24)$$

If

$$u_n=\frac{1}{(1-a^2)(1-a^4)\dots(1-a^{2n})}$$

$$v_n=\frac{1}{(1-a)(1-a^2)(1-a^3)\dots(1-^{2n})}\,a^{n(2n-1)}$$

Then $\quad u_n+u_{n-1}v_1+u_{n-2}v_2+\dots+n_1v_{n-1}+v_n$

$$=\frac{1}{(1-a)(1-a^2)(1-a^3)\dots(1-a^{2n})}.$$

(*i*) Let

$$w_n=\frac{1}{(1-a)(1-a^2)\dots(1-a^{2n})}.$$

The recurrent

$$\begin{vmatrix} 1 & 1 & 0 & 0 & \dots & 0 \\ w_1 & v_1 & 1 & 0 & \dots & 0 \\ w_2 & v_2 & v_1 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \end{vmatrix}_{n+1}$$

$$=u_n$$

and the recurrent

$$\begin{vmatrix} 1 & 1 & 0 & 0 & \dots & 0 \\ v_1 & w_1 & 1 & 0 & \dots & 0 \\ v_2 & w_2 & w_1 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \end{vmatrix}_{n+1}$$

$$=\frac{1}{(a^2-1)(a^4-1)\dots(a^{2n}-1)}a^{n(n-1)}$$

(*ii*)

$$E_3=\begin{vmatrix} 1 & 1 & 0 \\ \frac{1}{(1-a^2)(1-a^3)} & \frac{1}{1-a^2} & 1 \\ \frac{1}{(1-a^2)(1-a^3)(1-a^4)(1-a^5)} & \frac{1}{(1-a^2)(1-a^4)} & \frac{1}{1-a^2} \end{vmatrix}$$

$$E_n=(-1)^{n-1}\frac{1}{(1-a^2)(1-a^3)(1-a^4)\dots(1-a^{2n-1})}a^{n(2n+1)}$$

The proof is similar to that given for (23).

Hence

$$\frac{1+\frac{1}{(1-a^2(1-a^3)}x+\frac{1}{(1-a^2)(1-a^3)(1-a^4)(1-a^5)}x^2+\dots}{1+\frac{1}{1-a^2}x+\frac{1}{(1-a^2)(1-a^4)}x^2+\dots}$$

$$=1+\frac{1}{(1-a^2)(1-a^3)}a^4x+\frac{1}{(1-a^2)(1-a^3)(1-a^4)(1-a^5)}a^{10}x^2+\dots$$

$$+\frac{1}{(1-a^2)(1=a^3)\dots(1-a^{2n+1})}a^{n(2n+1)}x^n+\dots \qquad \dots \quad (25)$$

Relations and recurrents, like those obtained from (24) can also be obtained from (25).

If

$$v_n=\frac{1-a}{1-a^{n+1}}w_n$$

and

$$s_n = \frac{1-a}{1-a^{n+1}}\, a^{2n} v_n$$

then $$s_n + s_{n-1}u_1 + s_{n-2}u_2 + \dots\dots u_n = r_n,$$

$$\begin{vmatrix} 1 & 1 & 0 & 0 & \dots & 0 \\ r_1 & s_1 & 1 & 0 & \dots & 0 \\ r_2 & s_2 & s_1 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & 0 \end{vmatrix}_{n+1}$$

$$= u_n$$

and

$$\begin{vmatrix} 1 & 1 & 0 & 0 & \dots & 0 \\ s_1 & r_1 & 1 & 0 & \dots & 0 \\ s_2 & r_2 & r_1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \end{vmatrix}_{n+1}$$

$$= \frac{1}{(a^2-1)(a^4-1(\dots(a^{2n}-1)}\, a^{n(n-1)}$$

(*ii*) Let

$$d_4 = \begin{vmatrix} 1 & a-1 & 0 & 0 \\ a & a^2-1 & (a-1)(a^2-1) & 0 \\ a^3 & a(a^3-1) & (a^2-1)(a^3-1) & (a-1)(a^2-1)(a^3-1) \\ a^6 & a^2(a^4-1) & a(a^3-1)(a^4-1) & (a^2-1)(a^3-1)(a^4-1) \end{vmatrix}$$

$$= (a-1)(a^2-1)(a^3-1) \begin{vmatrix} 1 & a-1 & 0 \\ a & a^2-1 & (a-1)(a^2-1) \\ a^3 & a(a^3-1) & (a^2-1)(a^3-1) \end{vmatrix}$$

$$= (a-1)(a^2-1)(a^3-1)d_3$$

$$= (a-1)^3(a^2-1)^2(a^3-1)$$

Similarly

$$\beta_{n+1} = (a-1)^n(a^2-1)^{n-1}\dots(a^{n-1}-1)^2(a^n-1)$$

Let

$$x_n=\frac{1}{(a^2-1)(a^4-1)\dots(a^{2n}-1)}\,a^{n(n-1)}$$

$$f_4=\begin{vmatrix} 1 & 1 & 0 & 0 \\ v_1 & x_1 & 1 & 0 \\ v_2 & x_2 & x_1 & 1 \\ v_3 & x_3 & x_2 & x_1 \end{vmatrix}$$

$$=\frac{1}{(a-1)^3(a^2+1)^3(a^3-1)^2(a\ \ -1)^2(a^5-1)(a^6-1)}$$

$$\times\begin{vmatrix} 1 & 1 & 0 & 0 \\ a & a-1 & (a-)(a^2-1) & 0 \\ a^6 & a^2(a-1)(a^3-1) & (a-1)(a^3-1)(a^4-1) & (a-1)(a^3-1)(a^3-1) \\ a^{15} & a^6(a-1)(a^3-1)(a^2-1) & a^2(a-1)(a^3-1)(a^5-1)(a^6-1) & (a-1)(a^3-1)(a^4-1)(a^5-1)(a^6-1) \end{vmatrix}$$

The last determinant becomes

$$=-(a-1)^2(a^3-1)\begin{vmatrix} 1 & a^2-1 & 0 \\ a^2(a-1) & 1-a^3-a^4+a^5 & (a^2-1)(a^4-1) \\ a^6(a-1)(a^3-1) & a^2(a^3-1)(1-a^5-a^6+a^7 & (a^4-1)(1-a^5-a^6+a_0 \end{vmatrix}$$

By $\text{row}_3-a^7\text{row}_2$, $\text{row}_2-a^3\text{row}_1$, the above determinant becomes

$$=-\begin{vmatrix} 1 & a^2-1 & 0 \\ a^2 & a^4-1 & (a^2-1)(a^4-1) \\ a^6(a-1) & a^2(1-a^3-a^6+a^7 & (a^4-1)(1-a^5-a^6+a^7 \end{vmatrix}$$

Similarly

$$f_n=(-1)^{n-1}\frac{1}{(a-1)(a^3-1)\ldots(a^{2n-3}-1)}$$

We have also

$$w_n+u_1w_{n-1}+u_2w_{n-2}a^2+\times+u_rw_{n-r}a^{r-1}+\times+u_na^{n-1}=v_n.$$

This again by row$_3-a^2$row, becomes

$$=-\begin{vmatrix} 1 & a^2-1 & 0 \\ a^2 & a^4-1) & (a^2-1)(a^4-1) \\ a^6 & a^2(1-a^6) & (a^4-1)(a^6-1) \end{vmatrix}=d_3(a^2).$$

Hence

$$f_4=-\frac{1}{(a-1)(a^2-1)\ldots(a^6-1)}.$$

VII. (*i*) Let

$$=\begin{vmatrix} 1 & 1 & 0 & 0 & 0 \\ a & a & a-1 & 0 & 0 \\ a^3 & a^2 & a(a^2-1) & (a-1)(a^2-1) & 0 \\ a^6 & a^3 & a^2(a^3-1) & a(a^2-1)(a^3-1) & (a-1)(a^2-1)(a^3-1) \\ a^{10} & a^4 & a^3(a^4-1) & a^2(a^3-1)(a^4-1) & a(a^2-1)(a^3-1)(a^4-1) \end{vmatrix}$$

By col$_2-$col$_1$, it becomes

$$=(1-a)\begin{vmatrix} a^2-a^3 & (a-1)(a^2-1) & \\ a^3-a^5 & a(a^2-1)(a^3-1) & (a-1)(a^2-1)(a^3-1) \\ a^4-a^6 & a^2(a^3-1)(a^4-1) & a(a^2-1)(a^3-1)(a^4-1) \end{vmatrix}$$

$$=(-1)^2(a-1)^2(a^2-1)^{\overline{2}}(a^3-1)^2a^n\begin{vmatrix} a & 1 & 0 \\ a^2 & a & a-1 \\ a^4 & a^2 & a(a^2-1) \end{vmatrix}$$

$$=(-1)^2(a-1)^2(a^{\overline{2}}-1)^2(a^2-1)^2a^{3+1}g_3$$

$$=(-1)^2(a-1)^4(a^2-1)^{\overline{2}}(a^3-1)^2a^6.$$

Similarly

$$g_{2n+1}=(a-1)^{2n}(a^2-1)^{2n-2}(a^3-1)^{2n-4}\ldots(a^{3n-2}-1)^2$$

$$(a^{3n-1}-1)^2a^{n+1} \text{ and } g_{3n}=0.$$

Hence

$$\frac{1+\frac{1}{a-1}ax+\frac{1}{(a-1)(a^2-1)}a^3x^2e\times}{1+\frac{1}{a-4}an+\frac{1}{(a-1)(a^2-1)}a^2x^23\times}$$

$$=1+\frac{1}{a^2-1}a^2x^2+\frac{1}{(a^2-1)(a^4-1)}a^6x^4c$$

$$+\frac{1}{(a^2-1)\times(a^{2n}-1)}a^{n(n+1)}x^{2n} \qquad \times \qquad (26)$$

Here it is to be observed that

$$\begin{vmatrix} 1 & 1 & 0 \\ b_1x & a_1x & 1 \\ b_2a^2 & a_2x^2 & a_1x \\ \times & \times & \times \end{vmatrix}_{n+1} = x^n\begin{vmatrix} 1 & 1 & 0 \\ b_1 & a_1 & 1 \\ b_2 & a_2 & a_1 \\ \times & \times & \times \end{vmatrix}_{n+1}$$

(*ii*) Let h_5

$$=\begin{vmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & a-1 & 0 & 0 \\ a & 1 & a^2-1 & (a-1)(a^2-1) & 0 \\ a^3 & 1 & a^3-1 & (a^2-1)(a^3-1) & (a-1)(a^2-1)(a^3-1) \\ a^6 & 1 & a^4-1 & (a^3-1)(a^4-1) & (a^2-1)(a^3-1)(a^4-1) \end{vmatrix}$$

$$=-(a-1)\begin{vmatrix} 1-a & (a-1)(a^2-1) & 0 \\ 1-a^2 & (a^2-1)(a^3-1) & (a-1)(a^2-1)(a^3-1) \\ 1-a^3 & (a^3-1)(a^4-1) & (a^2-1)(a^3-1)(a^4-1) \end{vmatrix}$$

$$=(-1)^2(a-1)\begin{vmatrix} a-1 & (a-1)(a^2-1) & 0 \\ a(a^2-1) & a(a^2-1)(a^2-1) & (a-1)(a^2-1)(a^3-1) \\ a^3(a^3-1) & a^2(a^3-1)(a^2-1) & a(a^2-1)(a^3-1)(a^3-1) \end{vmatrix}$$

$$=(-1)^2(a-1)^2(a^2-1)^2(a^3-1)^2g_2.$$

Similarly

$$g_n=(-1)^2(a-1)^2(a^2-1)^2\cdots(a^{n-2}-1)^2 g_{n-2}.$$

(*iii*) Let

$$i_4=\begin{vmatrix} 1 & 1 & 0 & 0 \\ a_1 & a & 1 & 0 \\ a_2 & a^2 & a & 1 \\ a_3 & a^3 & a^2 & a \end{vmatrix}$$

Mutiplying the third and the fourth columns respectively by a and a^2 and then dividing the second, third, and the fourth rows respectively by a, a^2, and a^3, we obtain

$$i_4=a^3\begin{vmatrix} 1 & 1 & 0 & 0 \\ \frac{a_1}{a} & 1 & 1 & 0 \\ \frac{a_2}{a^2} & 1 & 1 & 1 \\ \frac{a_3}{a^3} & 1 & 1 & 1 \end{vmatrix}$$

By col_2-col_3, col_3-col_4, then by row_4-row_3,

$$i_4=(-1)^3(a_3-aa_2)\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$i_{n-1}=(-1)^n(a_n-aa_{n-1})$$

Note :—The source of this paper is the following two theorems :—

(1) $$\frac{1+b_1x+b_2x^2+b_3x^3+\times}{1+a_1x+a_2x^2+a_3x^3+\times}$$

$$=1-\begin{vmatrix} 1 & 1 \\ b_1 & a_1 \end{vmatrix}x+\begin{vmatrix} 1 & 1 & 0 \\ b_1 & a_1 & 1 \\ b_2 & a_2 & a_1 \end{vmatrix}x^2-\times$$

and

$$(2) \qquad = \frac{1}{1+} \; \frac{a_1 x}{1+} \; \frac{a_2 x}{1+} \, .$$

(1) On a theorem in determinants. (2) On algebraic remainders. (3) On a theorem in simultaneous equations and (4) On the solution of a set of simultaneous equations.

By

Haripada Datta, M.A. (Edin.)

I. On a theorem in determinants. We have

$$\begin{vmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_1 \\ a_3 & c_3 & d_3 \end{vmatrix} \begin{vmatrix} b_1 & c_1 & f_1 \\ b_3 & c_2 & f_2 \\ b_3 & c_3 & f_3 \end{vmatrix} - \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} \begin{vmatrix} c_1 & d_1 & f_2 \\ c_2 & d_2 & f_2 \\ c_3 & d_3 & f_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & c_1 & d_1 & 0 & 0 & 0 \\ a_1 & c_2 & d_2 & 0 & 0 & 0 \\ a_3 & c_3 & d_3 & 0 & 0 & 0 \\ 0 & 0 & d_1 & b_1 & c_1 & f_1 \\ 0 & 0 & d_2 & b_2 & c_2 & f_2 \\ 0 & 0 & d_3 & b_3 & c_3 & f^3 \end{vmatrix} - \begin{vmatrix} a_1 & b_1 & c_1 & 0 & d_1 & 0 \\ a_1 & b_2 & c_2 & 0 & d_1 & 0 \\ a_3 & b_3 & c_3 & 0 & d_3 & 0 \\ 0 & 0 & 0 & c_1 & d_1 & f_1 \\ 0 & 0 & 0 & c_2 & d_2 & f_2 \\ 0 & 0 & 0 & c_3 & d_3 & f_3 \end{vmatrix}$$

(in these determinants five columns are common)

$$= - \begin{vmatrix} a_1 & b_1 & c_1 & 0 & d_1 & 0 \\ a_2 & b_2 & c_2 & 0 & d_2 & 0 \\ a_3 & b_3 & c_3 & 0 & d_3 & 0 \\ 0 & b_1 & 0 & c_1 & d_1 & f_1 \\ 0 & b_2 & 0 & c_3 & d_3 & f_2 \\ 0 & b_3 & 0 & c_3 & d_3 & f^3 \end{vmatrix}$$

(by $\text{row}_4-\text{row}_1$; $\text{row}_5-\text{row}_2$; $\text{row}_6-\text{row}_3$)

$$=-\begin{vmatrix} a_1 & b_1 & c_1 & d_1 & 0 & 0 \\ a_2 & b_2 & c_2 & d_2 & 0 & 0 \\ a_3 & b_3 & c_3 & d_3 & 0 & 0 \\ -a_1 & 0 & -c_1 & 0 & c_1 & f_1 \\ -a_2 & 0 & -c_2 & 0 & c_2 & f_1 \\ -a_3 & 0 & -c_3 & 0 & c_3 & f_3 \end{vmatrix}$$

by $(\text{col}_3+\text{col}_5)$

$$=-\begin{vmatrix} a_1 & b_1 & c_1 & d_1 & 0 & 0 \\ a_2 & b_2 & c_2 & d_2 & 0 & 0 \\ a_3 & b_3 & c_3 & d_3 & 0 & 0 \\ -a_1 & 0 & 0 & 0 & c_1 & f_1 \\ -a_2 & 0 & 0 & 0 & c_2 & f_2 \\ -a_3 & 0 & 0 & 0 & c_3 & f_3 \end{vmatrix}$$

$$=\begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix} \begin{vmatrix} a_1 & c_1 & f_1 \\ a_2 & c_2 & f_2 \\ a_3 & c_3 & f_3 \end{vmatrix}.$$

In the case of the determinants of order 4, we have

$$\begin{vmatrix} a_1 & c_1 & d_1 & f_1 & 0 & 0 & 0 & 0 \\ a_2 & c_2 & d_2 & f_2 & 0 & 0 & 0 & 0 \\ a_3 & c_3 & d_3 & f_3 & 0 & 0 & 0 & 0 \\ a_4 & c_4 & d_4 & f_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & f_1 & b_1 & c_1 & d_1 & g_1 \\ 0 & 0 & 0 & f_2 & b_2 & c_2 & d_2 & g_2 \\ 0 & 0 & 0 & f_3 & b_3 & c_3 & d_3 & g_3 \\ 0 & 0 & 0 & f_4 & b_4 & c_4 & d_4 & g_4 \end{vmatrix} - \begin{vmatrix} a_1 & b_1 & c_1 & d_1 & f_1 & 0 & 0 & 0 \\ a_2 & b_2 & c_2 & d_2 & f_2 & 0 & 0 & 0 \\ a_3 & b_3 & c_3 & d_3 & f_3 & 0 & 0 & 0 \\ a_4 & b_4 & c_4 & d_4 & f_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & f_1 & c_1 & d_1 & g_1 \\ 0 & 0 & 0 & 0 & f_2 & c_2 & d_2 & g_2 \\ 0 & 0 & 0 & 0 & f_3 & c_3 & d_3 & g_3 \\ 0 & 0 & 0 & 0 & f_4 & c_4 & d_4 & g_4 \end{vmatrix}$$

(of which seven columns are common)

$$=-\begin{vmatrix} a_1 & b_1 & c_1 & d_1 & f_1 & 0 & 0 & 0 \\ a_2 & b_2 & c_2 & d_2 & f_2 & 0 & 0 & 0 \\ a_3 & b_3 & c_3 & d_3 & f_3 & 0 & 0 & 0 \\ a_4 & b_4 & c_4 & d_4 & f_4 & 0 & 0 & 0 \\ 0 & b_1 & 0 & 0 & f_1 & c_1 & d_1 & g_1 \\ 0 & b_2 & 0 & 0 & f_2 & c_2 & d_2 & g_2 \\ 0 & b_3 & 0 & 0 & f_3 & c_3 & d_3 & g_3 \\ 0 & b_4 & 0 & 0 & f_4 & c_4 & d_4 & g_4 \end{vmatrix}$$

(by $\text{row}_5-\text{row}_1$; $\text{row}_6-\text{row}_2$; $\text{row}_7-\text{row}_3$; $\text{row}_1-\text{row}_4$: and then by $\text{col}_3+\text{col}_6$; $\text{col}_4+\text{col}_7$)

$$=\begin{vmatrix} b_1 & c_1 & d_1 & f_1 \\ b_2 & c_2 & d_2 & f_2 \\ b_3 & c_3 & d_3 & f_3 \\ b_4 & c_4 & d_4 & f_4 \end{vmatrix}\begin{vmatrix} a_1 & c_1 & d_1 & f_1 \\ a_2 & c_2 & d_2 & f_2 \\ a_3 & c_3 & d_3 & f_3 \\ a_4 & c_4 & d_4 & f_4 \end{vmatrix}$$

In general, if $|\, a_{11}a_{11}(n-4a_{1\,n-1}a_{1n} \,|$ denotes a determinant of order n, [where $(n-4)$ stands for the $(n-4)$ consecutive elements, $a_{13}, a_{14}, \ldots, a_{1\,n-2}$, of the first row] then

$$|\, a_{11}(n-4)a_{1\,n-1} \,|\;|\, a_{12}(n-4)a_{1n} \,|-|\, a_{11}a_{12}(n-4) \,|\;|\, (n-4)a_{1\,n-1}a_{1n}$$
$$=|\, a_{12}(n-4)a_{1\,n-1} \,|\;|\, a_{11}(n-4)a_{1n} \,|.$$

II. On algebraic remainders.

Let

$$f(n)=a_nx^n+a_{n-1}x^{n-1}+\ldots+a_1x+a_0$$

When $f(x)$ is divided by $x+\gamma_1$, the quotient

$$\phi_1(\gamma_1 x)=a_nx^{n-1}+(\gamma_1a_n+a_{n-1})x^{n-2}+(\gamma_1^2a_n+\gamma_1a_{n-1}+a_{n-2})x^{n-3}+\ldots$$
$$\ldots+(\gamma_1^{n-1}a_n+\gamma_1^{n-2}a_{n-1}+\ldots+\gamma_1a_2+a_1)$$

and the remainder

$$R_1(\gamma_1)=f(\gamma_1)$$

Again if $\phi_1(\gamma_1x)$ be divided by $x-\gamma_2$, the quotient

$$\phi_2(\gamma_1\gamma_2x)=a_nn^{n-2}+\{(\gamma_1+\gamma_2)a_n+a_{n-2}\}x_{n-3}+\ldots$$

and the remainder

$$R_2(\gamma_1\gamma_2)=\phi_1(\gamma_1\gamma_2)=a_n\gamma_2^{n-2}+(\gamma_1 a_n+a_{n-1})\gamma_2^{n-3}+\dots$$

$$=\frac{\gamma_1^n-\gamma_2^n}{\gamma_1-\gamma_1}a_n+\frac{\gamma_1^{n-1}-\chi_2^{n-1}}{\gamma_1-\gamma_2}a_{n-1}+\dots$$

$$+\frac{\gamma_1-\gamma_2}{\gamma_1-\gamma_2}a_1+\frac{1-1}{\gamma_1-\gamma_2}a_0$$

$$=\frac{f(\gamma_1)-f(\gamma_2)}{\gamma_1-\gamma_2}=\frac{\begin{vmatrix} f(\gamma_1) & f(\gamma_2) \\ 1 & 1 \end{vmatrix}}{\begin{vmatrix} \gamma_1 & \gamma_2 \\ 1 & 1 \end{vmatrix}}$$

Thus $$R_3(\gamma_1\gamma_2\gamma_3)=\frac{\phi_1(\gamma_1\gamma_2)-\phi_1(\gamma_1\gamma_3)}{\gamma_2-\gamma_3}$$

$$=\frac{1}{\gamma_2-\gamma_3}\left[\frac{\begin{vmatrix} f(\gamma_1) & f(\gamma_2) \\ 1 & 1 \end{vmatrix}}{\begin{vmatrix} \gamma_1 & \gamma_2 \\ 1 & 1 \end{vmatrix}}-\frac{\begin{vmatrix} f(\gamma_1) & f(\gamma_3) \\ 1 & 1 \end{vmatrix}}{\begin{vmatrix} \gamma_1 & \gamma_2 \\ 1 & 1 \end{vmatrix}}\right]$$

$$=\frac{\begin{vmatrix} f(\gamma_1) & f(\gamma_2) & f(\gamma_3) \\ \gamma_1 & \gamma_2 & \gamma_3 \\ 1 & 1 & 1 \end{vmatrix}}{\begin{vmatrix} \gamma_1^2 & \gamma_2^2 & \gamma_3^2 \\ \gamma_1 & \gamma_2 & \gamma_3 \\ 1 & 1 & 1 \end{vmatrix}}.$$

Generally $R_m(\gamma_1\gamma_2\dots\gamma_m)$

$$\begin{vmatrix} f(\gamma_1) & f(\gamma_2)\dots f(\gamma_m) \\ \gamma_1^{m-2} & \gamma_1^{m-2}\dots\gamma_m^{m-2} \\ \gamma_1^{m-3} & \gamma_2^{m-3}\dots\gamma_m^{m-3} \\ \dots & \dots \\ \gamma_1 & \gamma_2\ \dots\ \gamma_m \\ 1 & 1 \quad 1 \end{vmatrix} \div \begin{vmatrix} \gamma_1^{m-1} & \gamma^{m-1}\dots\gamma_m^{m-1} \\ \gamma_1^{m-2} & \gamma_2^{m-2}\dots\gamma_m^{m-2} \\ \dots & \dots \\ \gamma_1 & \gamma_1\ \dots\ \gamma_m \\ 1 & 1 \quad 1 \end{vmatrix}.$$

Hence evidently $\phi_{m-1}(\gamma_1\gamma_2\dots\gamma_{m-1}x)$ is obtained by putting x for γ_m in R_m.

If some of the quantities γ's are equal, the expressions for the remainders or quotients, at first sight, seem to be indeterminate, but as a matter of fact, they are not so.

Suppose $\gamma_2=\gamma_1+h$, where $h=0$.

Now by Taylor's expansion and subtracting the first column from the second, we have $R_4(\gamma_1\gamma_2\gamma_3\gamma_4)$

$$=h\begin{vmatrix} f(\gamma_1) & f'(\gamma_1) & f(\gamma_3) & f(\gamma_4) \\ \gamma_1^2 & 2\gamma_1 & \gamma_3^2 & \gamma_4^2 \\ \gamma_1 & 1 & \gamma_3 & \gamma_3 \\ 1 & 0 & 1 & 1 \end{vmatrix} \div h\begin{vmatrix} \gamma_1^4 & 3\gamma_1^2 & \gamma_3^3 & \gamma_3^3 \\ \gamma_1^2 & 2\gamma_1 & \gamma_3^2 & \gamma_4^2 \\ \gamma_1 & 1 & \gamma_3 & \gamma_4 \\ 1 & 0 & 1 & 1 \end{vmatrix}$$

Again, if also, $\gamma_3=\gamma_-+k$, where $k=0$,

$$R_4(\gamma_1\gamma_2\gamma_3\gamma_4)$$
$$=hk^2\begin{vmatrix} f(\gamma_1) & f'(\gamma_1) & \tfrac{1}{2}f(\gamma_1) & f(\gamma_4 \\ \gamma_1^2 & 2\gamma_1 & 1 & \gamma_4^2 \\ \gamma_1 & 1 & 0 & \gamma_4 \\ 1 & 0 & 0 & 1 \end{vmatrix} \div hk^2\begin{vmatrix} \gamma_1^3 & 3\gamma_1^2 & 3\gamma_1 & \gamma_4^3 \\ \gamma_3^2 & 2\gamma_1 & 1 & \gamma_4^2 \\ \gamma_1 & 1 & 0 & \gamma_4 \\ 1 & 0 & 0 & 1 \end{vmatrix}$$

III. On a theorem in simultaneous equations.

If a's are integers and if the determinant is a multiple of

$$\begin{vmatrix} a_1 & a_2 & \times & a_n \\ b_1 & b_2 & \times & b_n \\ c_1 & c_2 & \times & c_n \\ \times & \times & \times & \times \end{vmatrix}_n$$

for all integral values of

$$\begin{vmatrix} u_0 & u_1 & u_2 & \times & u_n \\ a_0 & a_1 & a_2 & \times & a_n \\ b_1 & b_1 & b_2 & \times & b_n \\ \times & \times & \times & \times & \times \end{vmatrix}_{n+1}$$

u's then the n—simultaneous—equations

$$\left.\begin{array}{ll} a_0+a_1x_1+a_2x_2+\times+a_nx_n=0 & \\ b_0+b_1x_1+\times & +b_nx_n=0 \\ c_0+c_1x_1+\times & +c_nx_n=0 \\ \times & \times \end{array}\right\}$$

will be satisfied for the integral values of the variables.

Suppose

$$\begin{vmatrix} u_0 & u_1 & u_2 & \times & u_n \\ a_0 & a_1 & a_2 & \times & a_n \\ b_0 & b_1 & b_2 & \times & b_n \\ \times & \times & \times & \times & \times \end{vmatrix}_{n+1} = f(v_0 u_1 u_2 - h_n) \begin{vmatrix} a_1 & a_2 & \times & a_n \\ b_1 & b_2 & + & b_n \\ c_1 & c_4 & \times & c_n \\ \times & \times & \times & \times \end{vmatrix}_n$$

Here it is to be seen that $f(u_0, u_1 - u_n)$ is a linear function in u's and the co-efficients of u's are all integers.

With the references within my possession, I am not in a position to do proper justice to the paper.

IV. On the solution of a set of n simultaneous equations.

To solve the simultaneous equations

$$x_1 + x_2 + x_3 + \cdots + x_n + 1 = 0$$

$$\frac{u_1+b_1}{u_1+a_1}x_1 + \frac{u_2+b_1}{u_2+a_1}x_2 + \times + \frac{n_n+a_1}{u_n+a_1}x_n + \frac{u_{n+2}+b_1}{u_{n+1}+a_1} = 0$$

$$\frac{(u_1+b_1)}{(u_1+a_1)}u_2 + \frac{(u_2+b_1)(u_2+b_2)}{(u_2+a_1)(u_2+a_2)}x_2 + \times + \frac{(u_{n+2}+b_1)(n_{n+1}+b_2)}{(u_{n+1}+d_2)(u_{n+1}+a_2)} = 0$$

$$\times \qquad \times \qquad \times$$

Let C_4

$$= \begin{vmatrix} 1 & 1 & 1 & 1 \\ \dfrac{u_1+b_1}{u_1+a_1} & \dfrac{u_2+b_1}{u_2+a_1} & \dfrac{u_3+b_1}{u_3+a_1} & \dfrac{u_4+b_1}{u_5+a_1} \\ \dfrac{(u_1+b_1)(u_1+b_2)}{(u_1+a_1)(u_1+a_2)} & \dfrac{(u_1+b_1)(u_2+b_2)}{(u_1+a_1)(u_2+a_2)} & \dfrac{(u_3+b_1)(u_3+b_2)}{(u_3+a_1)(u_3+a_2)} & \dfrac{(u_4+b_1)(u_4+b_2)}{(u_4+a_3)(u_4+a_2)} \\ \dfrac{(u_1+b_1)\times(u_1+b_4)}{(u_1+a_1)\times(u_1+a_3)} & \dfrac{(u_2+b_1)\times(u_2+b_3)}{(u_2+a_1)\times(u_2+a_3)} & \dfrac{(u_3+b_1)\times(u_3+a_3)}{(u_3+a_3)\times(u_3+a_3)} & \dfrac{(u_4+b_1)\times(u_4+b_2)}{(u_4+a_1)\times(u_4+a_3)} \end{vmatrix}$$

By the operations

(1) $\text{row}_4 - \text{row}_3 = (b_3 - a_3)\text{row}'_4$,

$\text{row}_3 - \text{row}_2 = (b_2 - a_2)\text{row}'_3$,

$\text{row}_2 - \text{row}_1 = (b_1 - a_1)\text{row}'_2$.

(2) $\text{row}'_4 - \text{row}'_3 = (b_2 - a_2)\text{row}''_4$,

$\text{row}'_3 - \text{row}'_2 = (b_1 - a_1)\text{row}''_3$.

(3) $\text{row}''_4 - \text{row}''_3 = (b_1 - a_1)\text{row}'''_4$,

we have $C_4 = (b_3 - a_3)(b_2 - a_2)^2 (b_1 - a_1)^3$

$$\times \begin{vmatrix} 1 & 1 & 1 & 1 \\ \dfrac{1}{u_1+a_1} & \dfrac{1}{u_2+a_1} & \dfrac{1}{u_3+a_1} & \dfrac{1}{u_4+a_1} \\ \dfrac{1}{(u_1+a_1)(u_1+a_2)} & \dfrac{1}{(u_2+a_1)(u_2+a_2)} & \dfrac{1}{(u_3+a_1)(u_3+a_2)} & \dfrac{1}{(u_4+a_1)(a_4+x_2)} \\ \dfrac{1}{(u_1+a_1)\times(u_1-a_3)} & \dfrac{1}{(u_2+a_1)\times(u_2+a_3)} & \dfrac{1}{(u_3+a_1)\times(u_3+a_3)} & \dfrac{1}{(u_4+a_1)\times(u_4+a_3)} \end{vmatrix}$$

Let this determinant be denoted by B_4.

By

(1) $\text{row}_4 + (a_1 - a_3)\text{row}_3 = \text{row}'_4$,

$\text{row}_3 + (a_1 - a_2)\text{row}_2 = \text{row}'_3$,

(2) $\text{row}'_4 + (a_2 - a_3)\text{row}'_3 = \text{row}''_4$,

we obtain

$$B_4 = \frac{1}{(a_1-a_2)(a_2-a_3)(a_2-a_3)}$$

multiplied by

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ \frac{1}{u_4+a_1} & \frac{1}{u_2+a_1} & \frac{1}{u_3+a_1} & \frac{1}{u_4+a_1} \\ \frac{1}{u_1+a_2} & \frac{1}{u_2+a_2} & \frac{1}{u_3+a_2} & \frac{1}{u_4+a_2} \\ \frac{1}{u_1+a_3} & \frac{1}{u_3+a_3} & \frac{1}{u_3+a_3} & \frac{1}{u_4+a_3}; \end{vmatrix}$$

This last determinant is due to Joachimsthal.

Notices respecting New Books presented to the Library.

1. Introduction to The Theory of Fourier Series and Integrals. By H. S. Carslaw. Second Edition, completely revised, pp. 1-323. 30/- net. (Macmillan & Co., Limited, 1921.)

This book forms the first volume of the new edition of Prof. Carslaw's *Fourier's Series and Integrals and the Mathematical Theory of the conduction of Heat* which was published in 1906 and was for some time out of print. In view of the extensive advance that has been made in recent years in the Theory of Fourier Series and Integrals and their applications to Heat Conduction, the issue of this excellent treatise in two volumes which has enabled the author to incorporate in the present volume some of the latest developments of the subject will be welcomed by all mathematical readers. The volume begins with a historical introduction and the first three Chapters deal with the convergence of infinite series and integrals and the development of the idea of a limit and a function founded upon the modern theory of real numbers. In Chapter IV, the Definite Integral is treated from Riemann's point of view and in Chapters V. and VI. are discussed the theory of series whose terms are functions of a single variable, and the theory of integrals which contain an arbitrary parameter. Chapter VII gives a treatment of Fourier's Series depending on Dirichlet's Integrals in which a prominent place has been given to Fejér's work, both in the proof of the fundamental theorem and in the discussion of the nature of convergence in the Fourier Series. Chapter IX is devoted to Gibbs's Phenomenon, and the last Chapter to Fourier's Integrals in which the work of Pringsheim has been used. Two appendices are added. The first deals with *Practical Harmonic Analysis and Periodogram Analysis* and in the second a bibliography of the subject is given. Lebesgue's theory of integration and the functions which this theory has introduced into the Theory of Fourier's Series and Integrals have been considered too advanced by the author for the treatise. It would however have been very useful if in the text or in in an Appendix the theory had been defined and its bearing on the Theory of Fourier Series and Integrals had been pointed out as this would have prepared the grounds for the students who would desire to pursue this fruitful development of the subject. The book will be found to be very useful by all students

who want to have a comprehensive grasp of the subject but the price of the book, namely 30 shillings net, will be considered too high by the Indian readers at the present rate of exchange.

2. Géométrie Synthétique des Unicursales. By E. Bally. Pp. 1-98. (Gauthier-Villars et Cie, Paris, 1920).

Contents: 1. Preface, 2. General Survey of the Cyloids. 3. Tangentral properties of the hypocycloids with three points of inflexion. 4. Hycycloid as inverse to a circle inscribed in the triangle formed by the points of inflexion. 5. Various properties of hypocycloids. 6. The cubical developments of its tangents. 2. Notes.

3. Lecons sur les Fonctions Automorphes. By G. Giraud. Pp. 1-123. (Guthier Villers et Cie, Paris, 1920.)

Contents: 1. Historical Introduction. 2. General Properties of certain automorphic groups. 3. The linear groups. Groups of Poincáre, of M. Picard, and if M. Fubini. 4. Quadratic groups comprising the hyperabelian groups. 5. Groups formed by several other, prolongeants. 6. The functions of Poincare. 7. Appendix.

4. Les spectres Numeriques By Michael Petrovitch with a preface by Emile Borel. Pp. 1-107, (Guthier Villars et Cie, 1919.)

This book has been very severely criticised by Prof. G. H. Hardy.

5. Cours de Cinématique Théorique, By H. Lacaze. (Guthier Villars et Cie, Paris, 1920.)

An elementary treatise dealing with, (1) vectors, (2) kinematic of a pont, (3) motion of a soid, distribution of velocities, (4) composition of accelerations, (5) displacement of vectors lying on a plane, together with supplements on these subjects giving fuller details.

6. Précis de Calcul Géométrique. By R. Leveugle. With a preface by M. H. Fehr. pp. 1-400. (Guthier Villars et Cie, Paris, 1920.)

The present volume gives an interesting treatment of vectors and quaternions and their applications to geometry, kinematics, Newtonian potential, motion of solids, theory of elasticity and the electromagnetic theory of light.

7. Table Des Nombres Premiers et de la décomposition des Nombres de 1 a 100000. By G. Inghirami. Pp. 1-35. Preliminary Explanation, pp. I—VIII. (Guthier Villars et Cie, Paris, 1919.)

8. Reflexions sur la Métaphysique du Calcul Infinitesimal. By L. Carnot. First Volume, pp. 1-117, Second Volume, pp. 1-105. (Guthier-Villers et Cie, Paris, 1921.)

The vclumes deal with the fundamental principles of infinitite-simal analysis as embodied in the differential and integral calculus and calculas of variations.

9. Elements de Géométrie. By A. C. Clairant. First Volume, pp. 1-95. Second Volume, pp. 1-103. (Guthier Villars et Cie, Paris, 1920.)

10. Mémoir sur La Chaleur. Par MM.Lavoisier et De Laplace. pp. 1-74. With two plates. (Guthier Villars et Cie, Paris, 1920.)

11. Géométrie et Analyse des Intégrales doubles. By A. Buhl. pp. 1-67. Scientia, No. 36. (Guthier-Villars et Cie, Paris, 1920.)

12. Bulletin de L' Institut Aerodynamique De Koutchino, Fascicule VI, Gauthier-Villars, Paris, 1920. The volume contains the following papers of interest to students of hydrodynamics and applied mathematics generally—

D. P. Riabouchinski: I. Sur les de glissement dans les fluides. II. Cinematique des movements discontinus dans les fluides. III. Sur la résistance de l'air aux grandes vilerses. IV. Sur l'antorotation des projectiles. V. De l' influence du vent dans le reperage par le son. VII. Théorie des fusées. VIII. Considérations générales sur les hélices. IX. Sur les séries de Fourier.

13. Oeuvres de G. H. Halphen publiées par les soins de C. Jordon, H. Poincaré, E. Picard, avec la collaboration de E. Vessiot. Tome I, II and III (Guthier-Villars, Paris.)

Tome I contains 39 papers mainly on Geometry, integration of linear equations and Probabilities. Pp. 1-567.

Tome II contains a portrait of Halphen and 42 papers mainly on Geometry, differential equations, elliptic functions Fourier and Abel series. Pp. 1-560.

Tome III contains the following four memories of Halphen :—(1) Memoir on the reduction of linear differential equations to integrable forms. (2) Memoir on the classification algebraic curves. (3) On some linear differential equations of the fourth order. (4) On the invariants of the linear differential equations of the fourth order. pp. 1-514. (Gauthier Villars, Paris, 1921.)

14. Oeuvres completes de Thomas Jan Stieltjes publiées par les soins de la Société Mathématique D' Amsterdam. Tome I and II.

Tome I contains a portrait of Stieltjes as frontispiece and 47 papers on various subjects of pure and applied mathematics.

Tome II contained 37 papers. (P. Nordhoff, Groningen.)

15. The Fundamentals, Vol. I. By Jogendrakumar Sengupta. pp. 1-85.

The author gives in his own way a discussion of the fundamental principles involved in (1) the Euclidean axioms, (2) the first postulate of Euclid, (2) the tenth axiom, and (3) Euclid's second postulate.

16. A criticism of Mr. R. D. Oldham's memoir on the structure of the Himalayas and of the Gangetic Plain as elucidated by Geodetic observations in India. By Lieut. Colonel H. McC. Cowie, R. E. Published by the Govt. of India, 1921.

S. K. B.

NOTES AND NEWS.

At the meeting of the Society held on the 26th September, 1921, the Society extended its heartiest greetings and warmest felicitations to M. G. Mittag-Leffler, emeritus Professor of the University of Stockholm and an Honorary Member of the Society on his completing the 75th year age in March last.

Nature has published a special number (vol. 106, No. 2677) February 17, 1921) devoted entirely to discussions of various aspects of relativity theory. It contains papers by Professor A. Einstein, Mr. E. Cunningham, Sir F. Dyson, Dr. A. C. D. Crommelin, Dr. C. E. St. John, Prof. G. B. Mathews, Mr. J. H. Jeans, Prof. H. A. Lorentz, Sir O. Lodge, Prof. H. Weyl, Prof. A. Eddington, Dr. N. Campbell, Miss. Dorothy Wrench, Mr. H. Jefferys, and Prof. H. W. Carr.

The *Journal de Mathématiques pures et appliquees* which was founded in 1836 by J. Liouville and is now edited by C. Jordon has entered a new phase in its long career. The Managing Committee formed by some of the most distinguished French Mathematicians including Professors Appel, Borel, Picard, Hadamard, Goursat, Painlevé, Lebesgue. and others are thinking of establishing the Journal on broader lines and have invited the English-speaking mathematicians to contribute to it. We appeal to the Indian Universities and all Indian Mathematicians who take an interest in advanced mathematical work that they may usefully subscribe this journal which is one of of the best known among the Mathematician. The first number of the fourth volume of the eighth series for the year 1921 of this journal opens with an obituary notice of Professor Georges Humbert of the Ecole Polytechnique and the Collège de France, member of the Paris Academy of Sciences in the section of geometry, who died January 22, 1921 at the age of sixty-two years and also contains the following papers :—(1) Sur les formes d' Hermite ternaires dans un corps quadratique imaginaire (champs $\sqrt{-1}$ et $\sqrt{-2}$); par M. Georges Humbert. (2) Transformations of surfaces applicable to a quadric, by L. P. Eisenhart. (3) Sur quelques questions de minimum, relatives aux courbes arbiformes et sur leurs rapports avec le calcul des variations ; par M. Henri Lebesgue.

At the meeting of the Calcutta Mathematical Society held on September 26, 1921, the following papers were read :—(1)

Mr. Brajendranath Chakravarti: "On the distortion of the *rings and brushes* as observed through a twin crystal." (2) Mr. Panchanan Das: "On the disturbed electron orbits in electromagnetic field. (3) Mr. Satischandra Chakravarty: "On the transformation of a general determinant into a continuent and a recurrent and a new method of solving simultaneous equations." (4) Pandit Oudh Upadhyaya: (*i*) On the failure of Legendre's rule in a problem in the theory of nmbers. (*ii*) On a problem considered by S. Ramanujan." (5) Prof. Sudhansukumar Banerjee: "On the dissipation of energy of a vibrating membrane." (6) Mr. Mohitmohan Ghosh: "On cyclic and asymptotic lines."

The following courses of instruction to be delivered in the Institut de Mathématiques de l'Université de Strasbourg have been sent to us by Prof. Fréchet and are published for the information of our students :—

Helped by the favourable exchange level, a number of students are coming to study in French Universities. It is important to bring to their attention that, since November, 1919, the University of Strasbourg completely reorganised, is working in full order. Its teaching staff is more than equal in number to what it was under German rule, and its equipment, already excellent in many respects has been greatly improved where it was deficient.

Concerning mathematical teaching, young students will be offered in Strasbourg the usual standard courses on Analysis, Mechanics, Astronomy, whose programme are permanent and should require the student's activity for two or three years. Further more, a set of *research* courses have been arranged for the use of candidates to the "Doctorat de l'Université de Strasbourg," and of scholars generally. (French diplomas are required for registration in view of the doctorship, but can be dispensed with on production of equivalent foreign diplomas, with appreciation of the student's ability by one of his former Professors).

Programme of *research* courses during the academic year 1921-1922.

First Semester (1st Nov., 1921—28th Feb., 1922).

Mathematical Physics : Mr. Bauer : Statistical theory of heat.

3 lectures a week.

Higher Analysis : Mr. Fréchet : Extension of the notions of integral and of number of dimensions to the theory of abstract sets.

3 lectures a week.

Second Semester (1st March, 1922—30th June, 1922).

Mathematical Physics: Mr. Bauer: Dynamical electricity and restricted relativity.

3 weekly lectures.

Higher Analysis: Mr. Fréchet: Interpolation and adjustment.

3 weekly lectures.

Hydrodynamics: Mr. Villat: On recent applications of usual transcendental functions (Legendre, Bessel—...) in physical mathematics.

3 weekly lectures.

Differential Geometry: Mr. Pérès: Minima surfaces.

2 weekly lectures.

Theory of Functions: Mr. Valiron: Study of an analytic function near an essential singularity.

2 weekly lectures.

For further information apply (in French or English) to M. le Directeur de l'Institut de Mathématiques de Strasbourg, Bas-Rhin, France.

Details concerning lodgings etc. will be supplied by the Comité de Patronage des étudiants étrangers Université, Strasbourg, Bas-Rhin, Frence.

Students who want to improve their knowledge of the French language during the vacation may apply to the last address for the tract on "Summer Courses," organised by the "Faculté des Lettres de Strasbourg to meet their wishes.

Altogether fifteen volumes of the great edition of Euler's *Opera Omnia* have been published by the Euler Commission of the Swiss Society of Naturalists and of these volumes five have been issued since the outbreak of the war. All these fifteen volumes have been received in the Society's Library.

18

ON THE SOLUTION OF THE EQUATION $\nabla^4\psi=0$ IN BIPOLAR CO-ORDINATES.

A Question of Priority

The partial differential equation of the fourth order $\nabla^4\psi=0$, where $\nabla^4=\nabla^2.\nabla^2$ and ∇^2 is the two dimensional Laplacian $\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$, is of fundamental importance in many problems. For the two dimensional slow steady motion of a viscous fluid, the stream function ψ satisfies the equation $\nabla^4\psi=0$. It was shown by Airy that in the two dimensional case the stresses may be derived by partial differentiations from a single stress function and that in the absence of body forces, this stress function χ (say) satisfies the equation $\nabla^4\chi=0$.

The two dimensional solutions of $\nabla^4\psi=0$ have been obtained in several systems of curvilinear co-ordinates. The solutions in Cartesian co-ordinates are well-known. The solutions in polar co-ordinates were given by Michel. An outline of the solution in elliptic co-ordinates is given in Love's Elasticity.

The solutions of the equations $\nabla^4\psi=0$ in bipolar co-ordinates (ξ,η) for which the co-ordinate curves are co-axial circles and which are defined by

$$x+iy=c\tan\tfrac{1}{2}(\xi+i\eta),$$

or by

$$\xi+i\eta=\log\frac{x+i(y+c)}{x+i(y-c)}$$

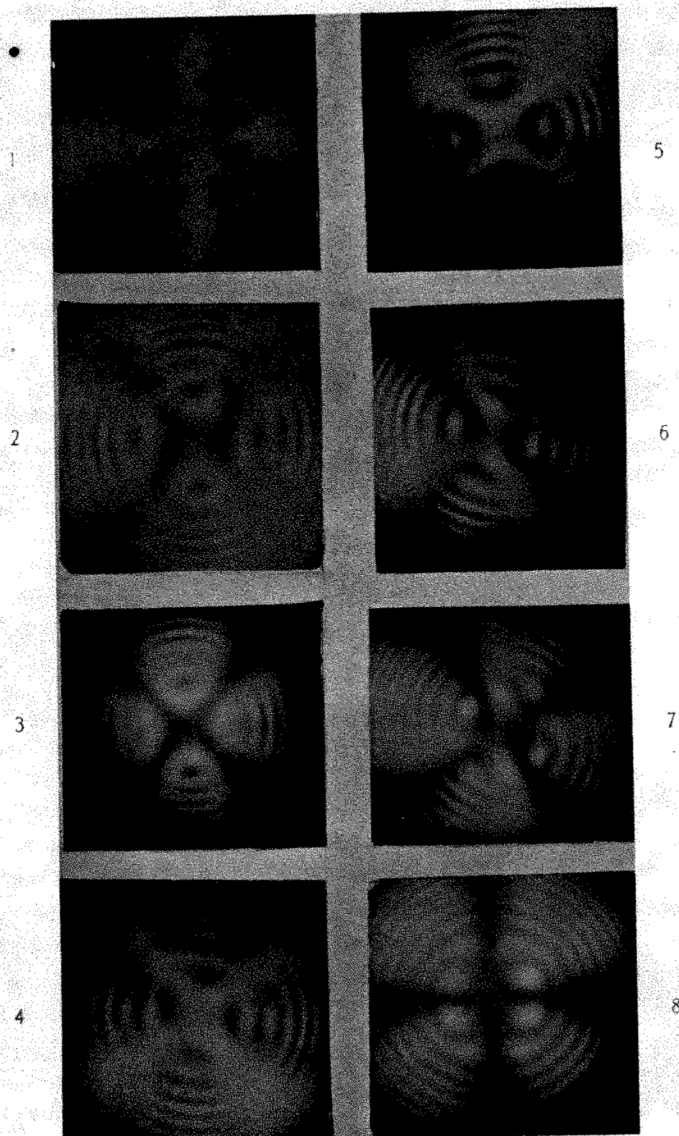
is of some importance in view of the applications that can be made of them in many physical problems. The solutions of this equation in the above system of co-ordinates were *first* given by Mr. Bijon Dutt, who worked as a research student under my direction and guidance. The solutions were given by him in connection with the problem of the steady motion of a viscous fluid due to the rotation of two rigid cylinders about parallel axes. His paper was read before the Calcutta

Mathematical Society, on March 9th, 1919 and was published in the Bulletin of the Calcutta Mathematical Society, Vol. X, No. 1 which was issued in June, 1919. The solutions are given in pages 54-58 of this issue of the Bulletin. This paper by Mr. Bijon Dutt is noticed in *Revue Semestrielle des publications mathématiques*, Vol. XXVIII, p. 13, 1919, April to October.

In the *Philosophical Transactions* of the Royal Society, Ser. A, Vol. 221, pp. 265-293, Dr. G. B. Jeffery gives exactly the same solutions of the same equation in bipolar co-ordinates in connection with the problem of plane strain and plane stress in these co-ordinates and writes as follows: "In this paper the complete solution (of $\nabla^4\psi=0$) is given for bipolar co-ordinates for which the co-ordinate curves are co-axial circles. This solution enables us to treat the problems of an infinite plate containing two circular holes, a semi-infinite plane bounded by a straight edge and containing one circular hole, and a circular disc with an eccentric circular hole." This paper by Dr. Jeffery was received by the Royal Society on May 15th, 1920 and published in November 8th, 1920. It would, however, be clear that I anticipated Dr. Jeffery when I suggested this problem to Mr. Bijon Dutt nearly a year and a half earlier and that the results obtained by Mr. Bijon Dutt were also published more than a year earlier than those of Dr. Jeffery's.

That the author of a memoir accepted for publication in the *Philosophical Transactions* of the Royal Society should be unaware of the results on identical matters published in the Bulletin of the Calcutta Mathematical Society in recent years of which a large number of copies are sent regularly, as each number is published, to the learned institutions, Universities and several mathematicians in the British isles is rather unfortunate. This is not the first occasion when I had to draw attention of mathematicians to a question of priority of this kind (see for instance my letter in *Nature*, No. 2423, Vol. 97, April 6th, 1916, p. 123).

S. K. Banerji.



Illustrating the distorted " Rings and Brushes " as observed through a twin Crystal.

19

THE DISTORTION OF THE "RINGS AND BRUSHES" AS OBSERVED THROUGH A "TWIN CRYSTAL."

BY

BRAJENDRANATH CHAKRABARTI.

[*Plate*]

1. INTRODUCTION.

Crystals are frequently found which are obviously of a composite character, that is, are composed of more than a single individual crystal of the same substance and in which the parts belonging to different individuals, are united in a definite and regular manner, the peculiar mode of union being characteristic of the substance.

The explanation of the formation of twin crystals is rather difficult. Lord Kelvin[1] however, has touched upon the problem. He has given a very lucid exposition of the whole process of crystal-building from a solution. In the particular class of crystals producing twins, he considers the constituent molecules symmetrical on the two sides of a plane passing through itself and also on the two sides of a plane perpendicular to this plane, that is to say, his crystalline molecules are egg-shaped. A real crystal which is growing by addition to a face would give layer after layer regularly. But, if *by some change in internal circumstances*, the molecules that would go to the formation of a layer are all oriented to 180°, with respect to the molecules that formed the previous layer, a twinning plane is formed and if the remaining layers form in the same way as the last mentioned one, then we shall get a crystal having two different parts separated by a twin-plane between. If, again, the process of orientation continues only to some layers and then due to *the re-establishment of the initial*

[1] Baltimore Lectures, Page 629, Part 37-38.

conditions, the layers form as at the start, we shall get two portions to the two sides with similarly oriented molecules, enclosing a thin layer in which the molecules are turned through 180°. The case will resemble that of Iceland Spar. But the main difficulty in the explanation is to understand *what is that change in the circumstances*, that causes the crystalline molecules to turn through 180° in the process of crystal formation and how is it in the case of Iceland Spar that *the initial conditions are re-established* after a thin layer is formed under the changed conditions. The difficulty is still greater when we come to consider the case of repeated twinning as in the case of potassium chlorate crystals where it is found that a very large number of twinning layers may be formed with a most surprisingly regular periodicity and constancy of thickness. Thus there is a good deal in regard to the mechanism of formation of these twinning layers that is as yet but imperfectly understood.

2. Optical behaviour of a Spath Hemitrope.

On looking at a source of light through a twin crystal of Iceland Spar, generally it is found that three images of the source are formed; the central one of which is always stationary. If the source of light is an incandescent electric lamp, then the images are beautifully coloured, the nature of the colour changing with the orientation of the plane of the crystal to the incident beam. It is also observed that the two outside images are polarised in perpendicular planes. On rotating the crystal, the two outside images are found to rotate about the central one and in the course of a revolution there are positions in which both of them disappear, the remaining one becoming the most brilliant for the time.

The author has also found that when a beam of light is allowed to fall upon a crystal cut perpendicular to the axis and also polished in such a way that the lines, in which the twinning lamina cuts the planes of the main crystal, are on the surface, some interesting diffraction effects are observed. Fringes are noticed in the region of the transmitted light which show a remarkable asymmetry. The edges of the lamina also appear to be luminous. The phenomena appear to belong to the class of laminar diffraction effects but there are certain features regarding them of which the explanation is not clear. The author is at present engaged in a fuller study of these effects and the investigation mentioned above will probably prove of considerable interest in relation to the determination of the optical nature of the twinning layer.

On examining the crystal with convergent plane polarised light and between crossed nicols, the ordinary rings and brushes are found to be distorted, the amount and the nature of the distortion changing with the orientation of the crystal. However, is some cases, the distortion more or less, resembles that produced by the superposition of a quarter-wave mica-plate upon a crystal of Iceland-Spar cut perpendicular to the axis so that the plane containing the optic axes of the mica-plate make an angle of 45° with the vibration planes of the crossed nicols.

The explanation of the phenomena of refraction has been given in text books on Optics.[1] There the formation of the three images and the polarisation are all explained. The disappearance of the outside images in certain positions as stated before, is connected with the orientation of the optic axis of the intervening layer to that of the main crystal.

The problem of the form of the rings and brushes through a twin crystal has been treated mathematically, but they have all considered the case where a twinning plane separates two crystals. The problem of the two similar crystal wedges separated by a thin twinned lamina inserted between them in an inclined position appears not to have been solved and an investigation in this direction forms the subject matter of the present paper.

3. Explanation of the Photographs.

The crystal that was employed in the following experiments was cut perpendicular to the axis and the portions free from the twinning layer showed the ordinary "Rings and Brushes" peculiar to crystals cut in such a manner. The angle which the twinning plane makes with the faces of the crystal is determined by observing the marginal outline of the plane as seen on looking through the edge of the crystal.

All the photographs were obtained keeping the analyser and polariser fixed (crossed-position), and by rotation of the crystal in its plane. It is found that the changes repeat themselves four times during a turn of the crystal through 2π. More-over, the change in the nature of the rings due to the introduction of the thin twin layer is the

[1] Mascart-Traite D'Optique Vol. II, page 192.

most prominent near the central part of the system. Figs. 1 to 4 (Plate) represent the changes for a rotation of the crystal through $\pi/2$ when the analyser has its plane vertical and the polariser horizontal. In Fig 1, the crystal is placed so that the twinning layer extends vertically up and down so that its section with the faces of the crystal are vertical straight lines. Light is always allowed to fall normally upon the first face of the crystal in order to get rid of the disturbing effect of the three images formed by refraction. Figs. 2, 3, 4, are obtained on rotating the crystal in its own plane through $\pi/2$, when the system again correspond with Fig. 1, but rotated through 90°. As we proceed further beyond $\pi/2$, the Figs. 2, 3, 4 comes in the reverse order, Fig. 4 being followed by Figs. 3 and 2 and at π we get back to Fig. 1. On rotating further the figures 2, 3, 4 comes in turn similarly as in the first quadrant and the figure at $3\pi/2$ correspond with one at $\pi/2$. The order of succession of the system along the last quadrant correspond with that of the second quadrant. Figs. 5 to 8 were obtained by keeping the analyser and polariser in some position other than the crossed and at the same time rotating the crystal also. In general, the systems are cumbrous in these cases, and four cases are selected out as showing the change rather more systematically. Figs. 7 and 8 have the peculiarity that they are complementary in nature to Figs. 4 and 1 respectively.

4. Physical theory of Distortion.

Since the crystal may be considered to be composed of two crystals *viz.*, one the regular, cut perpendicular to the axis and the other the thin twin layer cut in a direction inclined to this optic axis, we may consider the total path-retardation δ in any direction as made up of $\delta_1+\delta_2$, where

δ_1 = path-retardation due to the main crystal,

δ_2 = path-retardation due to the twin layer.

The expression for path-retardation due to any thin crystal may be obtained in the form,

$$\frac{\delta}{T}=-\frac{\sqrt{V^2-a^2\sin^2 i}}{a}+\frac{(a^2-c^2)\sin\chi\cos\theta\sin i}{a^2\cos^2\chi+c^2\sin^2\chi}$$

$$\frac{\delta_1}{T}\frac{\sqrt{(a^2\cos^2\chi+c^2\sin^2\chi)(V^2-c^2\sin^2 i)-c^2(a^2-c^2)\sin^2\chi\cos^2\theta\sin^2 i}}{a^2\cos^2\chi+c^2\sin^2\chi},$$

where, a, b, c, are the axes of the ellipsoid of elasticity,

V=Velocity of light in vacuo,

i=angle of incidence,

θ=azimuth of the plane of incidence with respect to the plane $\xi\zeta$ which contains the greatest axis oz of the ellipsoid of polarisation and $\chi=\angle Z0\zeta$.

Hence

$$\delta_1=\mu_0\left\{\frac{\sqrt{\mu_0^2-\sin^2 i}}{\mu_0}\right.$$

$$\text{and}\ \frac{\delta_2}{t}=\sqrt{\mu_2^2-\sin^2}$$

Hence, the curves representing the same path-retardation will not be equidistant from the centre as in the case of a regular crystal where we get the rings; but the distance of the points will be the greatest along $\theta=90°$, and the least along $\theta=0$. Hence the curves will lose their circular nature and become more or less elliptic.

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ON THE POLARISATION AND INTENSITY IN THE COMPLEX ZEEMAN-EFFECT.

BY

PANCHANAN DAS.

In a previous paper[1] the author showed how the complex *Zeeman*-effect obeying Runge's law, might be accounted for by an atomic model based on Sommerfeld's ellipse-verein. But the question of the polarisation and intensity-distribution of the components as well as the limit to the number of these components were not discussed there. An attempt to discuss these on Bohr's Correspondence Principie[2] will be made here.

Runge's law may be stated as follows :—

If a be the resolution of a normal triplet, the formula showing the complex-effect, is given by

$$\Delta\nu = \frac{q}{s} a,$$

where q and s are Runge's "number" and "denominator" respectively.

The formula deduced from theoretical considerations in the paper previously referred to, is

$$\Delta\nu = \frac{(m-n)}{s} a,$$

where s is the number of electrons in the ellipse-verein, and m, n are angular quantum-numbers in the final and initial orbits.

We had arbitrarily discarded[1] Sommerfeld's Principle, *viz.* $m-n = \pm 1$, for our particular atomic model, but no restriction was imposed on the limit to the value of $m-n$. We shall show that it has roughly a limit equal to s, although there might be a large number of exceptions to the rule.

As Sommerfeld has done, we shall suppose that during the emission of energy-quanta, the classical electromagnetic equations approximately hold good.

[1] Bulletin of Calcutta Mathematical Society Vol. XII; No. 2, 1921.

[2] Memoirs of the Copenhagen Academy, 1918.

We shall follow Rubinowicz[1] in calculating the moment of electromagnetic momentum radiated during an emission from the ellipseverein.

Maxwell's equations in free ether are :—

$$\frac{1}{c}\frac{\partial H}{\partial t}=-\text{curl } E, \frac{1}{c}\frac{\partial E}{\partial t}=\text{curl } H.$$

If we put $E=\text{curl}\,\text{curl } A$, where A is a vector, A satisfies the equation,

$$\frac{1}{c^2}\frac{\partial^2 A}{\partial t^2}=\nabla^2 A.$$

If we assume that the radiation is a monochromatic spherical wave, we may put

$$A=P\cdot\frac{e^{ikr}}{r},$$

where $P=p\cdot e^{-i\omega t}$ ... (1)

$$p=\begin{cases} p_x=ae^{i\alpha} \\ p_y=be^{\ \beta} \\ p_z=ce^{i\gamma} \end{cases}$$

and $K=\frac{\omega}{c}=\frac{2\pi}{\lambda}$.

It can then be shown that the moment of momentum (electromagnetic) radiated during an emission has the components,

$$N=\begin{cases} N_x=\frac{W}{\omega}\cdot\frac{2bc\sin(\gamma-\beta)}{a^2+b^2+c^2} \\ N_y=\frac{W}{\omega}\cdot\frac{2ca\sin(\alpha-\gamma)}{a^2+b^2+c^2} \\ N_z=\frac{W}{\omega}\cdot\frac{2ab\sin(\beta-\alpha)}{a^2+b^2+c^2} \end{cases}$$

where W is the total energy radiated during an emission,

If we choose axes properly, so that the X Y plane coincides with the "equivalent Oscillation-plane," the applitude c becomes$=0$, and then we can write

$$N=\frac{W}{\omega}\cdot\frac{2ab\sin\gamma}{a^2+b^2},$$

if we replace $\beta-\alpha$ with γ.

[1] Physik. Zeitschr. Bd. 19, 1918.

If p be the angular momentum of the system of electrons constituting the ellipse-verein, we found in the previous paper that

$$2\pi p = nh, \text{ or } p = \frac{nh}{2\pi}.$$

If the angular momentum lost by the system during radiation be Δp, we must equate this to N; thus

$$N = \Delta p = \frac{\Delta n \cdot h}{2\pi} = (m-n)\frac{h}{2\pi}. \qquad \dots \quad (2)$$

Again $W = s \cdot h\nu$ in conformity with our hypothesis that the number of energy-quanta radiated from a union of s electrons is s, and we may put $\omega = 2\pi\nu$. Thus,

$$N = \frac{s \cdot h}{2\pi} \cdot \frac{2ab \sin\gamma}{a^2+b^2}. \qquad \dots \quad (3)$$

Comparing (2) and (3) we see that

$$\Delta n = m - n = s \cdot \frac{2ab \sin\gamma}{a^2+b^2}.$$

Since $a^2+b^2 > 2ab$, this shows that the maximum value of $m-n$ is $\pm s$.

Of course this is admittedly a very crude calculation, as the vector P in (1) is hardly representative of the radiation from an ellipse-verein, So it is only to be expected, that quite as often as not, $m-n$ may be greater than s.

When the invariable plane of the system of electrons does not change during radiation, the oscillation-plane determined by the vector P is identical with this invariable plane. But if there is a change in inclination of the invariable plane, the oscillation-plane is determined by the difference vector $\Delta p = p_e - p_a$, where p_a, and p_e are the angular momenta vectorially represented by perpendiculars to the initial and final positions of the invariable plane respectively[1].

When $\Delta n = 0$, either a or b or γ equals zero, hence the light is unaffected by the magnetic field and is linear-poharised.

When $\Delta n = 1,2,3$, etc, the angle γ has then definite values and we would expect the light to be elliptically polarised; and as Δn approaches the value s, the angle γ approaches 90°, and the oscillation-ellipse becomes a perfect circle.

When the direction of observation is transverce to the fielo, some more or less arbitrary assumptions are necessary to acccount for the relative polarisations.

[1] See the figure in Somverfeld's Atom-bau and Spektrallinien.

Much in the same way as in the classical theory of Zeeman-effect, the electrical displacement-vector is resolved into the three components, along the line of observation, the direction of the field, and the direction perpendicular to both of these respectively. Then the component parallel to the field is unaffected by the same. We resolve the remaining two components into clockwise and counter clockwise circular vibrations, which will acquire a precession of positive and negative senses under the influence of the magnatic field, and therefore have frequencies of radiation $=\nu \pm \frac{a}{s}(m-n)$.

If we assume that for small values of Δn, the invariable plane undergoes a change of inclination, the oscillation-plane being perpendicular to the vector difference Δp, will be parallel to the magnetic field, and radiations of frequency for which Δn is thus small, will have components parallel to the field. When Δn is large, we suppose that the invariable plane is unaltered, hence the oscillation plane coincides with the invariable plane and is perpendicular to the field. Hence the components, for which Δn is large, will have their electric vector perpendicular to the magnetic field. Summing up the result, the central unaffected line and the lines close to it are generally polarised parallel to the field and the rest are polarised perpendicular to the field.

The question of intensity is however more obscure than that of polarisation. When a Quantum-radiation takes place, very little is known about the paths that the radiating electrons are describing at that moment. So we have to make very vague assumptions as regards the kinetics of electrons during radiation. We can then expand the coordinates of the system in trigonometric series and apply Bohr's Correspondence Principle to derive expressions for intensity.

It is found advisable to give a brief account of the dynamical methods that have come into use in Quantum-theory.

If $q_1\ q_2$...are the generalised coordinates and $p_1\ p_2$...the corresponding mamenta of a dynamical system, the definite integrals

$$J_k = \int p_k dq_k \text{ etc}$$

are called the phase-integrals.

If the system is conditionally periodic *i.e.*, if it has coordinates which oscillate between definite libration-limits, it is possible to choose these phase integrals as the impulse-coordinates, and certain other variables known in astronomy as the angle-coodinates as the conjugate position-coodinates, which conserve the canonical form of the Hamiltonian equation of motion. If S be the Hamilton-Jacobian Actionpotential,

it is possible to express S in terms of thec oordinate q's and the phase-integral J's.

Thus,

$$S=S(q_1\ q_2 ... J_1\ J_2 ...)$$

which gives

$$\delta S=\sum \frac{\partial S}{\partial q_k}\,\delta q_k+\sum \frac{\partial S}{\partial J_k}\,\delta J_k.$$

Writing $p_k=\dfrac{\partial S}{\partial q_k}$ and $\omega_k=\dfrac{\partial S}{\partial J_k}$, these reduce to

$$\delta(S-\sum\omega_k J_k)=\sum p_k\delta q_k-\sum J_k\delta\omega_k.$$

Hence the condition for contact-transformation from p, q to J, ω is fulfilled. Therefore Hamilton's canonical equations hold good for $J. \omega.$ *i.e.*

$$\frac{dJ_k}{dt}=-\frac{\partial H}{\partial \omega_k}\text{ and }\frac{d\omega_k}{dt}=\frac{\partial H}{\partial J_k}.$$

It can be shown that H is a pure function of the J's and independent of ω's and is constant during motion.

Hence $\dfrac{dJ_k}{dt}=0$, or $J_k=$constant.

$$\frac{d\omega_k}{dt}=\frac{\partial H}{\partial J_k}=\nu_k,\text{ say.} \qquad ... \quad (4)$$

$$\therefore \omega_k=\nu_k t+\delta_k. \qquad ... \quad (5)$$

It follows from the definition of a phase-integral that if S_e and S_a are the final and initial values of S when the coordinate q_k makes one complete circuit between its libration-limits, then

$$S_e-S_a=J_k$$

If we differentiate this partially with reference to J_k and J_l, we get

$$\omega^{k}{}_e-\omega^{k}_a=1,\ \omega^{l}{}_e-\omega^{l}{}_a=0,\ l\neq k.$$

k, e, u, i etc , are suffices at bottom,

This shows that when the coördinate q_k returns to its former value after a complete circuit, the corresponding angle variable increases by unity, while the rest remains unaltered. Conversely, when the angle-coordinate increases by unity, the coordinate q_k returns to its initial value. The *q's* are then periodic functions of the angle-coordinates of period unity.

It follows that the q's can be developed as a function of the ω's in Fourier's series of the following form :—

$$q=\Sigma C_{\tau_1\tau_2}\, e^{2\pi i(\tau_1\omega_1+\tau_2\omega_2+\ldots)}$$

$$=\Sigma C_{\tau_1\tau_2}\, e^{2\pi i(\tau_1\nu_1+\tau_2\nu_2+\ldots)t+2\pi i(\delta_1\tau_1+\ldots)}$$

If it were any mechanical or optical system, we might expect from it radiations of frequency given by

$$\nu=\tau_1\nu_1+\tau_2\nu_2+ \qquad \ldots \quad (6)$$

And the square of the co-efficient $C_{\tau_1\tau_2}$ would be a good measure of the intensity of radiation.

Substituting ν_k etc from (4), we have, since H=W=total anergy,

$$\nu=\frac{\partial W}{\partial J_1}\tau_1+\frac{\partial W}{\partial J_2}\tau_2+ \qquad \ldots \quad (7).$$

Sommerfeld has shown that the correspondence between the classical and quantum-theory of Radiation is that between a differential-and a difference-co-efficient.

We know that the system of electrons in Bohr's atom does not radiate energy in the stationary paths; energy-radiation takes place only during the transit from one stationary path to another. If ΔW be the difference of total energy between the intial and final paths, then we know that

$$\nu=\frac{\Delta W}{h},$$

as was shown in the previous paper.

We next split up ΔW into a set of partial differences ΔW_1 ΔW_2 etc, such that when a quantum-number n_k changes by Δn_k, the corresponding change in ΔW is ΔW_k. Thus,

$$\nu=\frac{\Delta W_1}{h}+\frac{\Delta W_2}{h}\ldots$$

Now $J_k=n_k\cdot h$, $\therefore \Delta J_k=h\cdot\Delta n_k$.

$$\therefore \nu=\frac{\Delta W_1}{\Delta J_1}\Delta n_1+\frac{\Delta W_2}{\Delta J_2}\Delta n_2+ \qquad \ldots \quad (8)$$

If we regard Δn_k as occupying the place of τ_k in (7), the analogy between (7) and (8) becomes evident. When the quantum-numbers,

themselves are very large, the difference-coefficients become differential coefficients in the limit and the analogy becomes an identity.

Now, for an atomic model based on the ellipse-verein, the frequency is given by

$$S \cdot h\nu = \Delta W.$$

$$\therefore \nu = \frac{\Delta W_1}{\Delta J_1} \cdot \frac{\Delta n_1}{s} + \frac{\Delta W_2}{\Delta J_2} \cdot \frac{\Delta n_2}{s} + \quad \dots \quad (9)$$

This suggests that the coordinates of the system admit of expansion in the form

$$q = C\tau_1\tau_1 \dots e^{\frac{2\pi i}{s}} (\tau_1\nu_1 + \tau_1\nu + \dots)t + \quad \dots \quad (10$$

This forces us to the conclusion that during the process of radiation, the form of the paths are not necessarily elliptic. Indeed that is only to be expected, because the excitation of energy-radiation must always be ascribed to some external influence, be it a thermal collision or the impact of an ion, etc. Hence it is only too probable, that the elliptic paths would be disturbed during radiation.

Now the application of Bohr's Correspondence Principle is generally carried out as follows:—We form the sum of the complex $x + \iota y$ for all the electrons and expand it in terms of the angle-variables. From this expansion we pick out those terms only, of which the order numbers τ_1, τ_2 etc are equal to the possible changes Δn_k etc in the qnantum-numbers. The squares of these co-efficients are taken as a measure of the intensity.

If the azimuth of the system be denoted by θ and the corresponding angle variable be $\omega_1 = \nu_1 t + \delta_1$, Bohr has shown that the vector $x + \iota y$ in the invariable does not plane contain $e^{2\pi 2\omega_1 t}$ $e^{2\pi\iota, 3\omega_1 t}$ etc, provided that θ is a cyclic coordinate. But if we assume that during Quantum-radiation, the azimuth θ ceases to be a cyclic coordinate due to external influences, then $\sum(x + \iota y.)$ may involve exponentials of multiples of $i\omega \cdot 2\pi$. We assume in conformity with the result (9) that the interactions of the s electrons are such that multiples of $i\omega \cdot \frac{2\pi}{s}$ are also present.

Thus

$$\sum(x + \iota y) = \sum B_{\tau_1\tau_2} e^{\frac{2\pi i}{s}} (\tau_1\omega_1 + \tau_2\omega_2 + \dots)$$

When the magnetic field is imposed, the whole system acquires a precession of magnitude $\frac{1}{2}\cdot\frac{e}{m}\cdot\frac{H}{c}$ in azimuth. Bohr has shown that if θ is cyclic, $x+\iota y$ can be then expended thus :—

$$\Sigma(x+\iota y)=\Sigma B_{\tau_1\tau_2}\, e^{2\pi i\left\{\left(\omega_1\pm\frac{1}{4\pi}\frac{e}{m}\cdot\frac{H}{c}t\right)+\tau_2\omega_2+\dots\right\}}.$$

We may then reasonably suppose[1] that when θ ceases to be cyclic, the expansion is

$$\Sigma(x+\iota y)=\Sigma B_{\tau_1\tau_2}\, e^{\frac{2\pi i}{s}\left\{\tau_1\left(\omega_1\pm\frac{1}{4\pi}\cdot\frac{e}{m}\cdot\frac{H}{c}t\right)+\tau_2\omega_2+\dots\right\}}.$$

It is then evident that lines corresponding to changes in frequency of magnitude

$$\frac{\tau_1}{s}\cdot\frac{1}{4\pi}\cdot\frac{e}{m}\cdot\frac{H}{c}$$

will spring up as a result of the imposition of the magnetic field. This can be easily interpreted as Runge's law.

With our previous notation $\tau_s=\Delta n_s=m_s-n_s$.

If $A(m_1m_2\dots)$ be the number of molecules in the state determined by the quantum-numbers m_1, m_2 etc, and if $C^{m_1,m_2\dots}_{n_1,n_2\dots}$ be the probability of transit from the state $(m_1m_2\dots)$ to $(n_1n_2\dots)$ then it is obvious that the intensity of of any component is well represented by

$$A(m_1m_2\dots)\,C^{m_1m_2\dots}_{n_1n_2\dots}\;B^2_{\tau_1\tau_2}.$$

When $\tau_1=1$, Bohr has shown that an approximate agreement with the classical theory may be gained by assuming C as proportional to the quantum-number which is undergoing a transit. But in our theory τ_1 may have s values other than unity and the probability factor C is thus governed by the unknown statistical laws of the dynamics of quanta. and no estimate of it can yet be made. However it may be remarked, that the non-appearance of some components corresponding to some definite values of τ peculiar to the particular line in question, might be due to the probability C of the corresponding transit being very small.

[1] This generalisation of course needs an independent proof. The agreement with observed results justifies it to some extent.

Expressions for the product of Bessel Functions in a series of Bessel Functions.

By

Abanibhusan Datta.

[*Read, August 14th, 1921.*]

1. It is well-known that the product of two Legendre functions in a series of Legendre functions was obtained by Adams[1] by a process of induction. The corresponding expressions for the preduct of two Bessel functions do not appear to have been given by any previous writer. The expressions are of some importance in many physical problems. For instance Prof. S. K. Banerji in his paper[2] "On the Radiation of Light from the boundaries of diffracting apertures" evaluated an integral of the type

$$I=\int_{R_1}^{R_2} J_0(Ar)J_1(Br)dr$$

by an approximate method. But by expressing the product of two Bessel functions in a series of Bessel functions, the value of the integral can be exactly determined.

The object of the present paper is to obtain the expansions of the product of Bessel Functions in series of Bessel Functions. The expansions have been obtained by two different methods. The first one depends on a method given by Neumann[3] for expressing any arbitrary function in a series of Bessel functions and is applicable when the orders of functions in the product are all integers. The second method is based on a result obtained by Schönholzer for expressing the product of two Bessel functions in an infinite power series of the argument and the expansion is valid whether the orders of the Bessel functions in the product be integers or fractions.

[1] Adams, Proc. Royal Soc., London, 1878.

[2] Banerji, Phil. Mag., Vol. XXXVII, January, 1919.

[3] Neumann, *Journal für Math.*, LXVII, p. 310 (1867).

I

2. Let $f(z)=J_\mu(z)\,J_\nu(z)$, where μ and ν are integers.

We know that for all values of n,

$$2J'_n = J_{n-1} - J_{n+1}.$$

Differentiating J_n with respect to z, m times successively and denoting $\frac{d^m}{dz^m} J_n(z)$ by $J_n^m(z)$, we have for all values of n,

$$2^m J_n^m(z) = J_{n-m}(z) - {}^mC_1 J_{n-m+2} + {}^mC_2 J_{n-m+4} \dots$$

$$+ (-)^r\, {}^mC_r J_{n-m+2r} \dots + (-)^m J_{n+m}. \qquad \dots \quad (A)$$

Now as $J_n(0)=0$, $n \neq 0$, and $J_0(0)=1$, it is to be noticed that $J_n^m(z)$ will be equal to zero unless J_0 occurs on the right hand side of (A), and J_0 can not come in so long as $m<n$.

Hence $J_\mu(0)=0$, $J'_\mu(0)=0 \dots J_\mu^{\mu-1}(0)=0$ $\;2^\mu J_\mu^\mu(0)=1$, $\;\therefore J_\mu^\mu(0)=\dfrac{1}{2^\mu}$.

$$2^{2r+\mu} J_\mu^{2r+\mu} = (-)^r\, {}^{2r+\mu}C_r.$$

Similar results are also true for J_ν.

Differentiating $f(z)$ n times successively, by means of Leibnitz Theorem, we have

$$f^n(z) = J_\mu J_\nu^n + {}^nC_1 J_\mu^1 J_\nu^{n-1} + \dots + {}^nC_r J_\mu^r J_\nu^{n-r} + J_\mu^n J_\nu \qquad \dots \quad (B)$$

Substituting the above values in (B), we have

$$f(0)=0,\ f'(0)=0\ldots f^{\mu}(0)=0\ldots f^{\mu+\nu-1}(0)=0.$$

$$f^{\mu+\nu}(0)={}^{\mu+\nu}C_{\mu}\,J_{\mu}^{\mu}(0)J_{\nu}^{\nu}(0)=\frac{{}^{\mu+\nu}C_{\mu}}{2^{\mu+\nu}}.$$

$$f^{\mu+\nu+1}(0)=0.$$

$$f^{\mu+\nu+2}(0)={}^{\mu+\nu+2}C_{\mu}J_{\mu}^{\mu}(0)J_{\nu}^{\nu+2}(0)+{}^{\mu+\nu+2}C_{\mu+2}J_{\mu}^{\mu+2}(0)J_{\nu}^{\nu}(0)$$

$$=-\frac{1}{2^{\mu+\nu+2}}\left[{}^{\mu+\nu+2}C_{\mu}\,{}^{\nu+2}C_{1}+{}^{\mu+\nu+2}C_{\mu+2}\,{}^{\mu+2}C_{1}\right]$$

$$=-\frac{1}{2^{\mu+\nu+2}}\,{}^{\mu+\nu+2}C_{\mu+1}\,{}^{\mu+\nu+2}C_{1}$$

$$f^{\mu+\nu+3}(0)=0.$$

$$\ldots\qquad\ldots\qquad\ldots\qquad\ldots\qquad\ldots\qquad\ldots$$

$$f^{\mu+\nu+2r}(0)={}^{\mu+\nu+2r}C_{\mu}J_{\mu}^{\mu}(0)J_{\nu}^{\nu+2r}(0)+$$

$${}^{\mu+\nu+2r}C_{\mu+2}J_{\mu}^{\mu+2}(0)J_{\nu}^{\nu+2r-2}(0)+\ldots$$

$$+{}^{\mu+\nu+2r}C_{\mu+2r}J_{\mu}^{\mu+2r}(0)J_{\nu}^{\nu}(0)$$

$$=(-)^{r}\frac{1}{2^{\mu+\nu+2r}}\Big[{}^{\mu+\nu+2r}C_{\mu}\,{}^{\nu+2r}C_{r}+$$

$${}^{\mu+\nu+2r}C_{\mu+2}\,{}^{\mu+2}C_{1}\,{}^{\nu+2r-2}C_{r-1}+\cdots$$

$$+{}^{\mu+\nu+2r}C_{\mu+2r}\,{}^{\mu+2r}C_{r}\Big]$$

$$=(-)\;\frac{{}^{\mu+\nu+2r}C_{\mu+r}\;{}^{\mu+\nu+2r}C_{r}}{2^{\mu+\nu+2r}}.$$

$$f^{\mu+\nu+2r+1}(0)=0.\qquad \text{etc.}$$

Neumann has shown that

$$f(z)=\sum a_n J_n(z)$$

where

$$a_n=2^n\left[f^n(0)+\frac{n}{2^2\cdot 1!}f^{n-2}(0)+\frac{n(n-3)}{2^4\cdot 2!}f^{n-4}(0)+\dots\right].$$

Now substituting the values of $f(0)$, $f'(0)\dots f^n(0)$ obtained above, we find that

$$J_\mu(z)J_\nu(z)=\sum a_n J_n(z)$$

were

$$a_n=0, \text{ when } n \text{ is less than } \mu+\nu$$

and

$$a_{\mu+\nu+2r}=(-)^{r}\,{}^{\mu+\nu+2r}C_{\mu+r}\,{}^{\mu+\nu+2r}C_r$$

$$+(-)\frac{r-1}{(\mu+\nu+2r-2)}\,{}^{\mu+\nu+2r-2}C_{\mu+\nu-1}\,{}^{\mu+\nu+2r-2}C_{r-1}$$

$$+(-)^{r-2}\frac{(\mu+\nu+2r)\;(\mu+\nu+2r-3)}{2!}\,{}^{\mu+\nu+2r-4}C_{\mu+r-2}$$

$${}^{\mu+\nu+2r-4}C_{r-2}.$$

II

3. We shall now consider the expansion in the form

$$J_\mu(z)J_\nu(z)=\sum_n a_n J_{n+\mu+\nu}$$

when μ and ν may have any real value.

It has been proved by Schönholzer[1] that for all values of μ and ν,

$$J_\mu(z)J_\nu(z)=\sum_{m=0}^{\infty}\frac{(-)^m\,\Gamma(\mu+\nu+2m+1)\,(\tfrac{1}{2}z)^{\mu+\nu+2m}}{m!\,\Gamma(\mu+m+1)\Gamma(\nu+m+1)\Gamma(\mu+\nu+m+1)}$$

$$=\Sigma b_m\,(\tfrac{1}{2}z)^{\mu+\nu+2m}$$

$$\text{where } b^m=(-)^m\,\frac{\Gamma(\mu+\nu+2m+1)}{m!\,\Gamma(\mu+m+1)\Gamma(\nu+m+1)\Gamma(\mu+\nu+m+1)}.$$

$$\therefore \sum_{n=0}^{\infty} a_n J_{n+\mu+\nu}$$

$$\text{or} \sum_{n=0}^{\infty} a_n\Bigg[\left(\frac{z}{2}\right)^{n+\mu+\nu}\frac{1}{\Gamma(n+\mu+\nu+1)}\left\{1-\left(\frac{z}{2}\right)^2\frac{1}{n+\mu+\nu+1}\right.$$

$$\left.+\left(\frac{z}{2}\right)^4\frac{1}{(n+\mu+\nu+1)(n+\mu+\nu+2)}\cdots\right\}\Bigg]$$

$$=\sum_{n=0}^{\infty} a_m\,(\tfrac{1}{2}z)^{\mu+\nu+2m}.$$

Equating the co-efficients of z from above, we have

$$\frac{a_0}{\Gamma(\mu+\nu+1}=\frac{\Gamma(\mu+\nu+1)}{\Gamma(\mu+1)\Gamma(\nu+1)\Gamma(\mu+\nu+1)}$$

$$a_1=0$$

$$\frac{a_2}{\Gamma(\mu+\nu+3)}-\frac{a_0}{\Gamma(\mu+\nu+2)}=\frac{-\Gamma(\mu+\nu+3)}{1!\,\Gamma(\mu+2)\Gamma(\nu+2)\Gamma(\mu+\nu+2)}$$

$$a_3=0$$

etc.

[1] Schönholzer, Über die Auswertung bestimmter Integrale mit Hilfe von Veränderungen des Integrationsweges, p. 15: Bern, 1877. Also Nielsen's Cylinderfunktionen, p. 20.

$$\frac{a_{2r}}{\Gamma(\mu+\nu+2r+1)}-\frac{a_{2r-2}}{\Gamma(\mu+\nu+2r)}+\ldots+(-)^r\frac{a_0}{\Gamma(\mu+\nu+r+1)}$$

$$=\frac{(-)^r\qquad\Gamma(\mu+\nu+2r+1)}{r!\,\Gamma(\mu+r+1)\Gamma(\nu+r+1)\Gamma(\mu+\nu+r+1)}$$

It is easy to find the values of a_0, a_2, a_4, etc, from the above equations.

III

4. Now the series $\sum b_m(\frac{1}{2}z)^{\mu+\nu+2m}$ satisfies Cauchy's first test of convergence because

$$\frac{b_m(\frac{1}{2}z)^{\mu+\nu+2m}}{b_{m-1}(\frac{1}{2}z)^{\mu+\nu+2m-2}}=-\frac{(\mu+\nu+2m)(\mu+\nu+2m-1)(\frac{1}{2}z)^2}{m(\mu+m)(\mu+m)(\mu+\nu+m)}$$

and for all integral positive values of m is less than k, after a certain limit, where k is a definitely assigned proper fraction.

In this case the limit of $\dfrac{a_m J_{m+\mu+\nu}}{a_{m-2}J_{m+\mu+\nu-2}}$

is $\dfrac{b_m(\frac{1}{2}z)^{\mu+\nu+2m}}{b_{m-1}(\frac{1}{2}z)^{\mu+\nu+2m-2}}$ and the series $\sum a_m J_{m+\mu+\nu}$

is absolutely convergent.

My best thanks are due to Prof. S. K. Banerji for his suggesting this problem to me and also for the interest he has taken in my work.

ON THE TRANSFORMATION OF A GENERAL DETERMINANT INTO A CONTINUANT AND A RECURRENT AND A NEW METHOD OF SOLVING SIMULTANEOUS EQUATIONS.

BY

SATISCHANDRA CHAKRABARTI.

[*Read, Sept. 26th, 1921.*]

§ 1.

We shall first show how to transform a general determinant into a recurrent.

Let us take the rectangular array

$$\left|\begin{array}{cccccc} a_{1n+1}, & a_{1n}, & a_{1n-1} & \cdots & a_{11} & a_{11} \\ a_{2n-1}, & a_{2n}, & a_{2n-1} & \cdots & a_{22} & a_{21} \\ \cdots & \cdots & & \cdots & & \cdots \\ a_{nn+1}, & a_{nn}, & a_{nn-1}, & \cdots & \cdots & \cdots \end{array}\right|,$$

and let K_n be obtained from it by deleting the first column and ${}^{r}K_n$ by deleting the $(r+1)^{th}$ column.

Then we get the identity

$$\frac{a_{11}{}^{n}K_n}{K_n}-\frac{a_{12}{}^{n-1}K_n}{K_n}+\cdots(-1)^{n-1}a_{1n}\frac{{}^{1}K_n}{K_n}+(-1)^{n}a_{1n+1}=0 \ldots \quad (a_1)$$

Putting 1, 2, 3...for n in (a_1), we get the following identities

$$\frac{a_{n}{}^{n}K_{n}}{K_1}-a_{12}=0 \qquad \ldots \quad (a_2)$$

$$\frac{a_{11}{}^{2}K_2}{K_2}-\frac{a_{12}{}^{1}K_2}{K_2}+a_{13}=0 \qquad \ldots \quad (a_3)$$

$$\frac{a_{11}{}^{3}K_3}{K_3}-\frac{a_{12}{}^{2}K_3}{K_3}+\frac{a_{13}{}^{1}K_3}{K_3}-a_{14}=0. \qquad \ldots \quad (a_4)$$

$$\cdots \qquad \cdots \qquad \cdots \qquad \cdots \qquad \cdots \qquad \cdots$$

$$\frac{a_{11}{}^{n-1}K_{n-1}}{K_{n-1}} - \frac{a_{12}{}^{n-2}K_{n-1}}{K_{n-1}} + \ldots + (-1)^{n-1}a_{1n} = 0. \quad \ldots \quad (a_n)$$

We also have the equation

$$a_{11}A_{11} - a_{12}A_{12} + a_{13}A_{13} - \ldots (-1)^n a_{1\,n+1}A_{1\,n+1}$$
$$= (-1)^n K_{n+1} \ldots (a_{n+1})$$

where K_{n+1} is a general determinant and A_{11}, A_{12}, A_{13}...are respectively the minors of a_{11}, a_{12}, a_{13}...obtained from K_{n+1}.

Assuming K_{n+1} and all a's except a_{11} as unknown quantities and solving for K_{n+1} we get the general determinant

$$K_{n+1} = \text{the recurrent } (-1)^n a_{11} \begin{vmatrix} \frac{{}^1K_1}{K_1} & 1 & & & \\ \frac{{}^2K_2}{K_2} & \frac{{}^1K_2}{K_2} & 1 & & \\ \frac{{}^3K_3}{K_3} & \frac{{}^2K_3}{K_3} & \frac{{}^1K_3}{K_3} & 1 & \\ \ldots & \ldots & \ldots & \ldots & \\ \frac{{}^nK_n}{K_n}, & \frac{{}^{n-1}K_n}{K_n} & \ldots & \frac{{}^1K_n}{K_n} & 1 \\ A_{11} & A_{12} \ldots & A_{1n} & A_{1\,n+1} \end{vmatrix}$$

§ 2.

(*i*) We shall now show that the general determinant K_{n+1} may also be transformed into a continuant.[1]

$$\text{Let } S_{(1,2,\ldots h)} = (-1)^h\, a_{1n-h+1}A_{1n-h+1} + \ldots + (-1)^n\, a_{11}A_{11}$$

where $(1,2,\ldots h)$ at the root of s indicates that the expression contains all the terms, except the 1st, 2nd,...h^{th} of the left hand side of the equation $a_{1\,n+1}A_{1\,n+1} - a_{1n}A_{1n} + \ldots + (-1)^n a_{11}A_{11} = K_{n+1}$.

$$\text{Also let } P^{(r)}_{\substack{n\\(2,3,\ldots h)}} = \frac{(-1)^{r}\ {}^{r-1}K_n a_{1\,n+1}}{a_{1\,n+1}K_n + (-1)^h a_{1\,n-h+1}{}^hK_n + \ldots (-1)^n a_{1n}{}^nK_n}$$

[1] [*It was given by T. Muir in Trans. Roy. Soc., Edin., Vol. 30, p. 5*].

where $(2,3,\dots h)$ at the root of P denotes that the denominator of the fraction contains all the terms, except the 2nd, 3rd ... h^{th} of the identity

$$a_{1\,n+1}K_n - a_{1\,n}\,{}^1K_n + a_{1\,n-1}\,{}^2K_n - \dots + (-1)^n a_{1\,1}\,{}^nK_n = 0$$

and (r) at the top of P indicates that the numerator contains the determinant ${}^{r-1}K_n$ as a factor, where r is an integer intermediate between 1 and $h+1$.

Then we get the equation

$$\frac{K_{n+1}}{S_{(1,2)}} = a_{1\,n+1}\left(\frac{A_{1\,n+1}}{S_{(1,2)}} + \frac{1}{a_{1\,n+1}}\right) - a_{1\,n}\frac{A_{1\,n}}{S_{(1,2)}} \qquad \dots \quad (a_1)$$

and we also have

$$-a_{1\,n+1} + a_{1\,n}\,P^{(2)}_{n\,(2,3)} + a_{1\,n-1}\,P^{(3)}_{n\,(2,3)} = 0 \qquad \dots \quad (a_2)$$

$$-a_{1\,n} + a_{1\,n-1}\,P^{(2)}_{n-1\,(2,3)} + a_{1\,n-2}\,P^{(3)}_{n-1\,(2,3)} = 0 \qquad \dots \quad (a_3)$$

... ...

$$-a_{1\,3} + a_{1\,2}\,P^{(2)}_{2\,(2,3)} + a_{1\,1}\,P^{(3)}_{2\,(2,3)} = 0 \qquad \dots \quad (a_n)$$

$$-a_{1\,2} + a_{1\,1}\,P^{(2)}_{1\,(3)} = 0 \qquad \dots \quad (a_{n+1})$$

Assuming $a_{1\,2}, a_{1\,3}, a_{1\,4} \dots a_{1\,n+1}$ and K_{n+1} as unknown quantities and solving for K_{n+1}, we get

$\cdot\ K_{n+1}$ = the continuant

$$a_{11}\, S_{(1,2)} \left|\begin{array}{cccccccc}
\dfrac{A_{1n+1}}{S_{(1,2)}}+\dfrac{1}{a_{1n+1}}, & -\dfrac{A_n}{S_{(1,2)}} & & & & & & \\
-1 & \underset{(2,3)}{P^{(2)}_n} & \underset{(2,3)}{P^{(3)}_n} & & & & & \\
 & -1 & \underset{(2,3)}{P^{(2)}_{n-1}} & \underset{(2,3)}{P^{(3)}_{n-1}} & & & & \\
\cdots & \cdots & \cdots & \cdots & & & & \\
 & & & & -1 & \underset{(2,3)}{P^{(2)}_2} & \underset{(2,3)}{P^{(3)}_2} & \\
 & & & & & -1, & \underset{(2)}{P^{(2)}_1} &
\end{array}\right|$$

Let this continuant be called C_2.

(*ii*) Let us now take the equations

$$\frac{K_{n+1}}{S_{(1,2,3)}}=a_{1n+1}\frac{A_{1n+1}}{S_{(1,2,3)}}-a_{1n}\frac{A_{1n}}{S_{(1,2,3)}}$$

$$+a_{1n-1}\frac{A_{1n-1}}{S_{(1,2,3)}}+a_{11}\frac{1}{a_{11}} \qquad \ldots \quad (a_1)$$

$$-a_{1n+1}+\underset{(2,3,4)}{P^{(2)}_n}\, a_{1n}+\underset{(2,3,4)}{P^{(3)}_n}\, a_{1n-1}+\underset{(2,3,4)}{P^{(4)}_n}\, a_{1n-2}=0 \quad \ldots \quad (a_2)$$

$$-a_{1n}+\underset{(2,3,4)}{P^{(2)}_{n-1}}\, a_{1n-n}+\underset{(2,3,4)}{P^{(3)}_{n-1}}\, a_{1n-2}+\underset{(2,3,4)}{P^{(4)}_{n-1}}\, a_{1n-3}=0 \quad \ldots \quad (a_3)$$

$$\cdots \qquad \cdots \qquad \cdots \qquad \cdots \qquad \cdots \qquad \cdots$$

$$-a_{14}+\underset{(2,3,4)}{P^{(1)}_3}\, a_{13}+\underset{(2,3,4)}{P^{(3)}_3}\, a_{12}+\underset{(2,3,4)}{P^{(4)}_3}\, a_{11}=0 \quad \ldots \quad (a_{n-1})$$

$$-a_{13}+\underset{(2,3)}{P^{(2)}_2}\, a_{12}+\underset{(2,3)}{P^{(3)}_2}\, a_{11}=0 \qquad \ldots \quad (a_n)$$

$$-a_{12}+\underset{(2)}{P^{(2)}_1}\, a_{11}=0. \qquad \ldots \quad (a_{n+1})$$

Assuming $a_{12}, a_{13}, \ldots a_{1\,n+1}$ and K_{n+1} as the unknown quantities and solving for K_{n+1}, we get

$$K_{n+1} = a_{11}\, S_{(1,2,3)} \left| \begin{array}{cccccccc} \dfrac{A_{1\,n+1}}{S_{(1,2,3)}}, & -\dfrac{A_{1n}}{S_{(1,2,3)}}, & \dfrac{A_{1\,n-1}}{S_{(1,2,3)}} & -0 & 0 & & \ldots & 0\ \dfrac{1}{a_{11}} \\ -1, & \underset{(2,3,4)}{P_n^{(2)}} & \underset{(2,3,4)}{P_n^{(3)}} & \underset{(2,3,4)}{P_n^{(4)}} & & & & \\ & -1, & \underset{(2,3,4)}{P_{n-1}^{(2)}} & \underset{(2,3,4)}{P_{n-1}^{(3)}} & \underset{(2,3,4)}{P_{n-1}^{(4)}} & & & \\ \ldots & \ldots & \ldots & \ldots & \ldots & & \ldots & \ldots \\ \vdots & & & & -1 & \underset{(2,3,4)}{P_3^{(2)}} & \underset{(2,3,4)}{P_3^{(3)}} & \underset{(2,3,4)}{P_3^{(4)}} \\ & & & & & -1 & \underset{(2,3)}{P_2^{(2)}} & \underset{(2,3)}{P_2^{(3)}} \\ & & & & & & -1 & \underset{(2)}{P_1^{(2)}} \end{array} \right|$$

Let us call this determinant C_3.

Similarly it may be shown that C_1, C_4, $C_5 \ldots C_n$ are all equal to the general determinant K_{n+1}.

Thus we get a series of determinants C_1, C_2, C_3, $C_4 \ldots C_n$ each of which is equal to K_{n+1}.

By multiplying the even columns and even rows (or odd columns and odd rows) by -1, it is easy to show that C_n is the same recurrent as given in Art 1.

§ 3.

Making use of all the equations, except the 1st, of Art 2, and solving for $a_{1\,n+1}$, we obtain the continuant

$$\begin{vmatrix} P^{(2)}_{n\,(2,3)} & P^{(3)}_{n\,(2,3)} & & & & & \\ -1 & P^{(2)}_{n-1\,(2,3)} & P^{(3)}_{n-1\,(2,3)} & & & & \\ & -1 & P^{(2)}_{n-2\,(2,3)} & P^{(3)}_{n-2\,(2,3)} & & & \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \\ & & & -1 & P^{(2)}_{3\,(2,3)} & P^{(3)}_{3\,(2,3)} & \\ & & & & -1 & P^{(2)}_{2\,(2,3)} & P^{(3)}_{2\,(2,3)} \\ & & & & & -1 & P^{(2)}_{1\,(2)} \end{vmatrix} = \frac{a_{1\,n+1}}{a_{11}}$$

Let this determinant be denoted by $D_{(2,3)}$. Then similarly as in Art 2, we can get a series of determinants $D_{(2)}$, $D_{(2,3)}$, $D_{(2,3,4)}$

$D_{(2,3,\ldots n+1)}$ each of which is equal to $\frac{a_{1\,n+1}}{a_{11}}$. The last of there determinants, *viz.*, $D_{(1,3,\ldots n+1)}$ is the recurrent[1]

$$\begin{vmatrix}
\frac{{}^1K_1}{K_1}, & 1 & & & & \\
\frac{{}^2K_2}{K_2}, & \frac{{}^1K_2}{K_2}, & 1 & & & \\
\frac{{}^3K_3}{K_3}, & \frac{{}^2K_3}{K_3}, & \frac{{}^1K_3}{K_3}, & 1 & & \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\frac{{}^{n-1}K_{n-1}}{K_{n-1}}, & \frac{{}^{n-2}K_{n-1}}{K_{n-1}}, & & \cdots & \frac{{}^1K_{n-1}}{K_{n-1}}, & 1 \\
\frac{{}^nK_n}{K_n}, & \frac{{}^{n-1}K_n}{K_n}, & & \cdots & \frac{{}^2K_n}{K_n}, & \frac{{}^1K_n}{K_n}
\end{vmatrix}$$

§ 4.

A new method to solve simultaneous equations.

Let us consider the following four equations

$$a_1x_1 + a_2x_1 + a_3x_3 + a_4x_4 + a_5 = 0$$

$$b_1x_1 + b_2x_2 + b_3x_3 + b_4x_4 + b_5 = 0$$

$$c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4 + c_5 = 0$$

$$d_1x_1 + d_2x_2 + d_3x_3 + d_4x_4 + d_5 = 0$$

Take the determinant $\begin{vmatrix} a_2 & a_3 & a_4 & a_5 \\ b_2 & b_3 & b_4 & b_5 \\ c_2 & c_3 & c_4 & c \\ d_2 & d_3 & d_4 & d_5 \end{vmatrix}$

[1] *See "On Some Symmetric Determinants" by Haripada Datta, Bulletin, Calcutta Math. Soc., Vol. XI, No. 4,* 1919-20, p. 235.

and on this, perform the operation $\mathrm{col}_4 + x_1\,\mathrm{col}_3 + x_3\,\mathrm{col}_2 + x_2\,\mathrm{col}_1$, then the determinant becomes

$$\begin{vmatrix} a_2 & a_3 & a_4 & -a_1x_1 \\ b_2 & b_3 & b_4 & -b_1x_1 \\ c_2 & c_3 & c_4 & -c_1x_1 \\ d_2 & d_3 & d_3 & -d_1x_1 \end{vmatrix}$$

Hence $$\begin{vmatrix} a_2 & a_3 & a_4 & a_5 \\ b_2 & b_3 & b_4 & b_5 \\ c_2 & c_3 & c_4 & c_5 \\ d_2 & d_3 & d_4 & d_5 \end{vmatrix} = -x_1 \begin{vmatrix} a_2 & a_3 & a_4 & a_1 \\ b_2 & b_3 & b_4 & b_1 \\ c_2 & c_3 & c_4 & c_1 \\ d_2 & d_3 & d_1 & d_1 \end{vmatrix}$$

Thus x_1 is determined.

The general case may be similarly treated.

23

On the Application of Mathieu Functions to Physical Problems.

By

Bholanath Pal.

1. Harmonic functions of elliptic and hyperbolic cylinders can be determined from the solution of Mathieu's differential equation.[1] As Prof. Whittaker[2] has remarked, this equation comes up for study after the hypergeometric equation has been disposed of. It is of higher type than the hypergeometric equation and its degenerate cases, and its solution presents difficulties which do not arise in connection with those equations. It has been studied by various investigators including Prof. Whittaker, Lindemann,[3] Stieltjes,[4] Floquet,[5] Maclaurin,[6] Watson,[7] Young,[8] Lindsay Ince[9] and others. It has been pointed out that it has interesting applications in many physical and astronomical problems, for instance in the problems of, (1) tidal waves in a cylindrical vessel with an elliptic boundary, (2) the scattering of electromagnetic waves by a wire of elliptic cross section, (3) certain forms of steady vortex motion in an elliptic cylinder, etc. It has also wide application, though in a different way, in Celestial Mechanics in the treatment of perturbations and oscillations about periodic orbits.[10] But none of these applications appears to have been worked out in detail by means of Mathieu functions which have received such a remarkable development in recent years in the hands of Professor Whittaker and his pupils.

[1] Mathieu, Liouville's Journal (2), XIII (1868), pp 137-208.
[2] Whittaker, Proc. Int. congress of Math., Cambridge, 1912, 1, pp. 366-371.
[3] Lindemann, Math. Ann., XXII, pp. 117-123.
[4] Stieltjes, Astronomische Nach., CIX.
[5] Floquet, Ann. de l'E'cole Normale Supérieure (2), XII (1883), pp. 47-88.
[6] Maclaurin, Trans. Camb. Phil. Soc., XVII, pp. 41-108.
[7] Watson, Proc. Edinburgh Math. Soc., XXXIII.
[8] Young, Proc. Edinburgh Math. Soc., XXXII.
[9] Lindesay Ince, Proc. Edinburgh Math. Soc., XXXIII.
[10] Burns: Ast. Nach. No. 2533, S. 193-204 (1883) and No. 2553, S. 129-132, (1884).

In the present paper I have pointed out how the functions which have been so far developed can be made use of to solve definite physical problems.

2. Mathieu's differential equation is

$$\frac{d^2u}{dz^2}+(a+16q\cos 2z)u=0.$$

In the equation, if we change the independent variable by the substitution $\cos^2 z=x$, we get the following transformed equation

$$x(1-x)\frac{d^2u}{dx^2}+\frac{1}{2}(1-2x)\frac{du}{dx}+\frac{1}{4}(a-16q+32qx)u=0. \quad \ldots \quad (1)$$

When "a" vanishes, the equation (1) can be very easily integrated. When $a=0$, it can be written as

$$\left\{4x(1-x)\frac{d}{dx}-(4x-2)\right\}\left\{\frac{d}{dx}-8q\right\}u=0.$$

Putting

$$\frac{du}{dx}-8qu=v$$

and integrating, we get

$$\log v+\tfrac{1}{2}\log(x-x^2)=c$$

i.e. $$v=\frac{A}{\sqrt{x-x^2}}.$$

Therefore $$\frac{du}{dx}-8qu=\frac{A}{\sqrt{x-x^2}}$$

or $$\frac{d}{dx}\left(ue^{-8qx}\right)=\frac{Ae^{-8qx}}{\sqrt{x-x^2}}. \qquad \therefore ue^{-8qx}=A\int\frac{e^{-8qx}}{\sqrt{x-x^2}}\,dx+B$$

i.e. $$u=Ae^{8qx}\int\frac{e^{-8qx}}{\sqrt{x-x^2}}\,dx+Be^{8qx}.$$

3. In physical (as distinguished from astronomical) problems the constant "a" in Mathieu's differential equation is to be chosen to be such a function of q that the equation possesses a periodic solution.

These periodic solutions, which, in Prof. Whittaker's notation, are written

$$ce_0(z,q),\ ce_1(z,q),\ ce_2(zq),\ldots$$

$$se_1(z,q),\ se_2(z,q),\ldots$$

are infinite in number, and according to Hilbert,[1] they may be regarded as the *eigenfunktionen* (or autofunctions) of a certain integral equation, of which the corresponding *eigenwerte* (or autovalues) are the values of "a" concerned.

Let $ce_m(z,q)$ and $ce_n(z,q)$ be any two of these functions, so that

$$\frac{d^2 ce_m(z,q)}{dz^2}+(a_m+16q\cos 2z)ce_m(z,q)=0,$$

and
$$\frac{d^2 ce_n(z,q)}{dz^2}+(a_n+16q\cos 2z)ce_n(z,q)=0,$$

where a_m and a_n are two functions of q, corresponding to $ce_m(z,q)$ and $ce_n(z,q)$ respectively.

Multiplying the first by $ce_n(z,q)$ and the second by $ce_m(z,q)$ and subtracting one from the other, we have

$$ce_n(z,q)\ \frac{d^2 ce_m(z,q)}{dz^2}-ce_m(z,q)\ \frac{d^2 ce_n(z,q)}{dz^2}+(a_m-a_n)ce_m(z,q)ce_n(z,q)=0.$$

Integrating between the limits $-\pi$ and π and writing $ce'_m(z,q)$,

$$ce_m''(z,q)\ldots\text{for } \frac{dce_m(z,q)}{dz},\ \frac{d^2 ce_m(z,q)}{dz^2},\ldots.$$

[1] Hilbert: Göttingen Nachrichten, 1904, S. 213-234.

There is some difference between the Hilbert integral equation and the integral equation given by Whittaker. The *kernel* of the former has a discontinuity, where as the latter has a continuous *kernel*.

We have

$$(a_m - a_n)\int_{-\pi}^{\pi} ce_m(z,q)ce_n(z,q)dz = \int_{-\pi}^{\pi}[ce_m(z,q)ce''_n(z,q) - ce_n(z,q)ce''_m(z,q)]dz$$

$$= \Big| ce_n(z,q)ce'_m(z,q) - ce_m(z,q)ce'_n(z,q) \Big|_{-\pi}^{\pi}$$

$$=0.$$

Thus

$$\int_{-\pi}^{\pi} ce_m(z,q)ce_n(z,q)dz^* = 0, \text{ when } m \neq n.$$

Now put $m = n + \partial n$, where ∂n is very small.

Therefore

$$\partial n \frac{da_n}{dn}\int_{-\pi}^{\pi} ce_m(z.q)ce_n(z,q)dz = \Big| ce_n(z,q)\left[\frac{d}{dz}ce_n(z,q) + \partial n \frac{d^2}{dndz}ce_n(z,q)\right]$$

$$-\left[ce_n(z,q) + \partial n \frac{d}{dn}ce_n(z,q)\right]\frac{d}{dz}ce_n(z,q)\Big|_{-\pi}^{\pi},$$

$$= \partial n \Big| ce_n(z,q)\frac{d^2}{dndz}ce_n(z,q)$$

$$-\frac{d}{dn}ce_n(z,q)\frac{d}{dz}ce_n(z,q)\Big|_{-\pi}^{\pi}.$$

* *cf.* Whittaker and Watson: Modern Analysis, second edition, p. 404, § 19·31.

This result has been given by Whittaker in his Modern Analysis, but the method is quite different.

So that when $m=n$, in the limit we have

$$\int_{-\pi}^{\pi}\left\{ce_n(z,q)\right\}^2 dz = \frac{1}{\frac{\partial a_n}{\partial n}}\left| ce_n(z,q)\frac{d}{dn}ce'_n(z,q)-ce'_n(z,q)\frac{d}{dn}ce_n(z,q)\right|_{-\pi}^{\pi}$$

These relations enable us to determine the co-efficients when an arbitrary function of z is expanded as a series of the functions $ce_0(z,q)$, $ce_1(z,q)$, $ce_2(z,q)$,....

That is to say, if, for a function $f(z)$, an expansion of the form

$$f(z)=a'_0\, ce_0(z,q)+a'_1 ce_1(z,q)+\ldots+a'_n ce_n(z,q)+\ldots$$

exists, and if it is permissible to integrate term-by-term between the limits $-\pi$ and π, then

$$a'_n=\frac{\frac{\partial a_n}{\partial n}}{\left| ce_n(z,q)\,\frac{\partial}{\partial n}\,ce'_n(z,q)-ce'_n(z,q)\,\frac{\partial}{\partial n}\,ce_n(z,q)\right|_{-\pi}^{\pi}}\int_{-\pi}^{\pi}ce_n(t,q)f(t)dt.$$

Similar results will be true for $se_n(z,q)$.

4. From Floquet's theory of linear differential equations, we learn that the solution of Mathieu's differential equation

$$\frac{d^2u}{dz^2}+(a+16q\cos 2z)u=0$$

is of the type

$$u=Ae^{\mu z}\phi_1(z)+Be^{-\mu z}\phi_2(z),$$

where A and B denote arbitrary constants, μ is a constant depending on the constants a and q of the differential equation, and $\phi_1(z)$ and $\phi_2(z)$ are periodic functions of z. In some special cases the constant μ vanishes and the solution then reduces to a purely periodic function of z; but in general μ is not always zero. Thus though we can know the general character of the solution, great difficulties are experienced in its actual analytical determination. It is very difficult to find μ in terms

of a and q. Hill's method of determining μ introduces infinite determinant into analysis and to find μ as a root of an infinite determinant becomes unmanagable. The determination of the solution in the above form is however of considerable importance in view of the fact that in physical problems we often require solutions which themselves or their space derivatives have to vanish at infinity. Prof. Whittaker[1] has recently given a method of obtaining solutions of Mathieu's equation in the above form by introducing a new parameter σ in place of a. He has obtained a solution

$$u = A\Lambda(z,\sigma,q) + B\Lambda(z,-\sigma,q),$$

where A and B are constants, $\Lambda(z,\sigma,q)$ is of the form $e^{\mu z}u(z)$, μ being expressed in terms of σ and the parameter q, and $u(z)$ a periodic function of z, and $\Lambda(z,-\sigma,q)$ is obtained from $\Lambda(z,\sigma,q)$ by merely changing σ to $-\sigma$. When σ has either of the special values $\frac{\pi}{2}$ or 0, μ vanishes, $\Lambda(z,\sigma,q)$ and $\Lambda(z,-\sigma,q)$ become identical and the solution reduces to the periodic solutions $ce_1(z,q)$ or $se_1(z,q)$.

A. W. Young[2] has given a method of constructing the general solution which reduces to $ce_n(z,q)$ or $se_n(z,q)$ when $\sigma = -\frac{\pi}{2}$ or 0. We shall denote these functions by

$$\Lambda_1(z,\sigma,q),\ \Lambda_2(z,\sigma,q), \ldots \Lambda_n(z,\sigma,q)\ldots.$$

In the case of the function which reduces to $ce_0(z,q)$, σ is not introduced and we leave that out from this series.

We have

$$\Lambda_1(z,\sigma,q) = e^{4qz\sin 2\sigma}\Big[\sin(z-\sigma) + q\sin(3z-\sigma) + q^2\Big\{3\sin 2\sigma\cos(3z-\sigma)$$
$$+\cos 2\sigma\sin(3z-\sigma) + \frac{1}{3}\sin(5z-\sigma)\Big\} + \ldots\Big]$$

[1] Whittaker, Proc. of the Edinburgh Math. Soc. Vol. 32 (1913-14).

[2] Young, Proc. of the Edinburgh Math. Soc. Vol. 32 (1913-14).

where

$$a=1+8q\cos 2\sigma-8q^2(1+2\sin^2 2\sigma)+\ldots$$

$$\Lambda_2(z,\sigma,q)=e^{4q^2 z\sin 2\sigma}\left[\sin(2z+\sigma)+q\left\{2\sin\sigma+\frac{2}{3}\sin(4z+\sigma)\right.\right.$$

$$\left.\left.+\frac{1}{6}q^2\sin(6z+\sigma)+\ldots\right.\right],$$

where

$$a=4-\left(\frac{16}{3}-32\sin^2\sigma\right)q^2+\ldots$$

$\Lambda_1(z,\sigma,q)$ and $\Lambda_2(z,\sigma,q)$ reduce to $ce_1(z,q)$ or $se_1(z,q)$ and $ce_2(z,q)$ or $se_2(z,q)$ respectively according as $\sigma=-\frac{\pi}{2}$ or 0.

In this way a solution $\Lambda_n(z,\sigma,q)$ is constructed, which reduces to $ce_n(z,q)$ or $se_n(z,q)$ respectively, according as $\sigma=-\frac{\pi}{2}$ or 0, by assuming for it an expression

$$e^{Nq^n z\sin 2\sigma}\left[\sin(nz-\sigma)+q\alpha_1(z)+q^2\alpha_2(z)+\cdots\right],$$

along with

$$a=n^2+q\cdot\beta_1(z)+q^2\cdot\beta_2(z)+\ldots$$

and substituting them in the differential equation.

The corresponding second solutions are obtained by merely changing σ into $-\sigma$ in the above formulae and we shall denote these solutions by

$$\Lambda_1(z,-\sigma,q),\ \Lambda_2(z,-\sigma,q),\ldots\Lambda_n(z,-a,q),\ldots.$$

Thus the most general solution of Mathieu's differential equation may be represented by

$$u=\sum_{n=1}^{\infty}\left\{A_n\Lambda_n(z,\sigma,q)+B_n\Lambda_n(z,-\sigma,q)\right\}.$$

5. We now consider the problem of diffraction of an incident wave system by a screen of finite width.

An incident system of rays, travelling in the direction of y negative can be represented apart from the time factor, by

$$\phi=e^{iky}.$$

Introducing real variables ξ,η defined by the complex equation

$$x+iy=h\cosh(\xi+i\eta),$$

so that

$$x=h\cosh\xi\cos\eta, \qquad y=h\sinh\xi\sin\eta,$$

we have

$$\phi=e^{ikh\sinh\xi\sin\eta}$$

i.e. $\phi=e^{k'\cos\xi'\cos\eta'}$, where $kh=k'$, and $i\xi=\frac{\pi}{2}-\xi'$, $\eta=\frac{\pi}{2}-\eta'$.

Now we have

$$e^{k'\cos\xi'\cos\eta'}=\sum_{n=1}^{\infty} A_n ce_n(\xi',q)ce_n(\eta',q)^*$$

where A_n is an arbitrary constant and $ce_n(\xi',q)$ and $ce_n(\eta',q)$ are Mathieu functions of n^{th} order, and $q=\frac{k'^2}{32}$.

From the above relation, we can easily obtain

$$\int_{-\pi}^{\pi} e^{k'\cos\xi'\cos\eta'} ce_n(\xi',q)d\xi'=A_n ce_n(\eta',q)\int_{-\pi}^{\pi}\Big[ce_n(\xi',q)\Big]^2 d\xi',$$

all other terms vanishing by the relation

$$\int_{-\pi}^{\pi} ce_m(\xi',q)ce_n(\xi',q)d\xi'=0, \quad \text{when } m\neq n.$$

Since even Mathieu functions satisfy a homogeneous integral equation of the form

$$G(\eta)=\lambda\int_{-\pi}^{\pi} e^{k\cos\eta\cos\theta}\, G(\theta)d\theta;\dagger$$

we get

$$A_n=\frac{1}{\lambda\int_{-\pi}^{\pi}[ce_n(\xi',q)]^2 d\xi'},$$

* Whittaker and Watson, Modern Analysis, (second edition). p. 405.

† Whittaker and Watson. Modern Analysis, (second edition), p. 402,

where λ is a known constant and

$$\int_{-\pi}^{\pi}[ce_n(\xi',q)]^2 d\xi'^* = \frac{1}{\dfrac{\partial a_n}{\partial n}}\left| ce_n(\xi',q)\frac{d}{dn}ce'_n(\xi',q) - ce'_n(\xi',q)\frac{d}{dn}ce_n(\xi',q)\right|_{-\pi}^{\pi};$$

thus A_n is determined.

Let us assume for the diffracted system of rays

$$\phi' = \sum_{n=1}^{\infty} A'_n \Lambda_n(\xi', -\sigma, q) ce_n(\eta', q),$$

where A'_r is an arbitrary constant, and $\Lambda_n(\xi', -\sigma, q)$ is a function representing the general solution of Mathieu's equation of the 2nd kind.

The constant A'_n will be determined from the boundary condition. If we take $\xi' = a$, as the boundary we get the case of diffraction by an elliptic cylinder, and we have

$$\left|\frac{\partial}{\partial n}(\phi+\phi')\right|_{\xi'=a} = 0$$

where ∂n is an element of the normal to the curved surface $\xi = a$ drawn inwards.

On the curved cylindrical surface, we have $\dfrac{\partial}{\partial n} = \dfrac{1}{B}\dfrac{\partial}{\partial \eta}$, where A and B are the functions occuring in the linear element $ds^2 = A^2 d\xi^2 + B^2 d\eta^2 + dz^2$.

Hence

$$\sum_{n=1}^{\infty} A_n \frac{\partial}{\partial \eta} ce_n(\eta', q) ce_n(a, q) - \sum_{n=1}^{\infty} A'_n \frac{\partial}{\partial \eta} ce_n(\eta', q) \Lambda_n(a, -\sigma, q) = 0.$$

Equating co-efficients of $\dfrac{\partial}{\partial \eta} ce_n(\eta', q)$ to zero, we obtain

$$A'_n = A_n \frac{ce_n(a,q)}{\Lambda_n(a,-\sigma,q)},$$

where A_n is known, and therefore A'_n can be determined from this relation.

ξ=const. represents the ellipse

$$\frac{x^2}{h^2\cosh^2\xi} + \frac{y^2}{h^2\sinh^2\xi} = 1.$$

* Given before.

Giving different values to ξ, we get a system of ellipses having their common foci at the pts. $(\pm h,0,0)$.

When $\xi=0$, the ellipse reduces to a str. line joining the foci $(\pm h,0,0)$. So that when $\xi=0$, the cylinder reduces to a rigid lamina whose section in the z-plane is the line joining the foci $(\pm h,0,0)$. We obtain in this case, the case of diffraction by a rigid lamina or a screen of finite width.

In this case

$$A'_n = A_n \lim_{a=\frac{\pi}{2}}{}^{\circ} \left[\frac{ce_n(a,q)}{\Lambda_n(a-\sigma,q)} \right].$$

6. To solve the problem of diffraction by a rectilinear slit, a problem which has been worked out by the method of infinite series of images by Schwarzschild[1] from Sommerfeld's treatment of the diffraction problem by a semi-inifinite screen, we expand the incident wave system in the form

$$\phi = e^{ik\sinh\xi\sin\eta} = \sum_{n=1}^{\infty} A_n \Lambda_n(\eta',-\sigma,q)\Lambda_n(\xi',-\sigma,q).$$

Now assume an expression for the diffracted wave system of the type

$$\phi' = \sum_{n=1}^{\infty} B_n \Lambda_n(\xi',-\sigma,q) ce_n(\eta',q)$$

which vanishes at infinity.

If the boundary is determined by $\eta'=\beta$, we have

$$\left[\frac{\partial}{\partial n}(\phi+\phi') \right]_{\eta'=\beta} = 0,$$

and we get the case of diffraction by a hyperbolic cylinder and the unknown constants are easily evaluated.

When $\eta=0$, *i.e.*, $\eta'=\frac{\pi}{2}$, we get the case of diffraction by a rectilinear slit.

[1] Math. Ann., Bd. 55 (1902).

Aryabhata's Method of Determining the Mean Motions of Planets.

By

P. C. Sengupta.

1. Introductory Remarks.

The writers of Indian Astronomical works are generally silent as to their method of finding the mean motions of planets. Aryabhata however, has left us something definite in his work on this point. The stanza from the Aryabhatiyam that is intended here to interpret rightly will, it is hoped, throw a flood of light on the history of Indian Astronomy. It seems that none of the commentators have been able to understand it rightly and must have confounded the scholars by their misinterpretations.

क्षिति रवि योगाद्दिनकृद् रवीन्दुयोगात् प्रसाधितश्चेन्दुः ।
शशिताराग्रहयोगात् तथैव ताराग्रहाः सर्व्वे : ।

Gola. 48.

2. The Stanza from the Aryabhatiyam.

This may be translated as follows :—

" The sun has been determined from the conjunction of the earth (*i.e.* the horizon) and the sun, and the moon; similarly all the ' star planets' (*i.e.* the five planets Mercury, Venus, Mars, Jupiter and Saturn) have been found from their conjunctions with the moon."

3. Method of Finding the year.

As to the motion of the sun, the stanza says that it was determined from the conjunction of the horizon and the sun. It is difficult to see what method exactly was followed by Aryabhata to find the solar year; but let us takes into account his figures. He says that the sun performs 4,320,000 revolutions in 1,577,917,500 days, or that the year consists of 365·25,868 days which was indeed meant to be the sidereal year. Very

probably it was determined by the observation of the distances of the sun from some bright star at intervals of 365 and 366 days. One of the methods of doing this seems to have been by observation of the time that elapsed between the risings (or settings) of a bright star and the sun. The other methods of Indian Astronomy *viz.*, observation of the noon shadow of the pnomon, or of the amplitude of the sun at rising or setting, would lead to the tropical year and not the sidereal year. We may therefore conclude that Aryabhata attempted to find the length of the year by observing the length of the period of heliacal rising or setting of fixed stars. The phrase 'the conjunction of the earth and the sun' would mean the number of daily risings of the sun in such a period.

4. The Method of Finding the Sidereal Month.

The second part of the stanza relates to the motion of the moon and says that it was found from the conjunction of the sun and moon, *i.e.*, from the synodic month, the length of which according to Aryabhata is 29 da. 12 h. 44 m. 2·285 secs, the modern length of this month is 29 da. 12 h. 44 m. 2·864 secs. (Its value as given in the old and modern Surya-siddhantas are respectively 29 da. 12 h. 44 m. 2.7 secs. and 29 da. 12 h. 44 m. 2·96 secs). This synodic month may also have been found from the mean period of heliacal risings of the moon.

The derived sidereal month is according to Aryabhata of 27 da. 7 h. 43 m. 12·11 secs. the modern value being 27 da. 7 h. 43 m. 11·545 secs. Thus it is possible to give a rational inperpretation to the first half of the stanza. We now turn to the second half of the stanza where it is said that the mean motion of the 'star planets' was determined from their periods of conjunction with the moon.

5. Moon's Periods of Conjunction with a superior Planet.

In this investigation we shall assume that the orbits of planets are all circles having the sun for their common centre and are all coplanar; further the moon's orbit is also supposed to be coplanar with them and having the earth for its centre. We shall also assume that all orbits are discribed with uniform speed.

Let E, M, S, and J be respectively the positions of the earth, moon, sun and a superior planet, the sun and the superior planet being in conjunction; let the circle represent the orbits of the different bodies; let E_1, M_1, J_1 be the positions of the earth, moon, and the superior planet at rth conjunction with the moon, which is supposed to have taken place after d days. Let A be the point of intersection of E, S, J,

with the moon's orbit; let us suppose that the moon takes a days to describe the arc AM. Let the sidereal month, the year and the sidereal period of the superior planet be respectively denoted by M, y and y' in mean solar days. Through E_1 let $E_1\sigma$ be drawn paralled to E, S, J. Again let θ and θ^1 be the angles described by the earth and the superior planet round the sun in d days and a and a^1 their mean distances from the sun,

$$\text{then the } \underline{|J_1 E_1 \sigma} = \tan^1 \frac{a \sin\theta + a^1 \sin\theta^1}{a\cos\theta + a^1 \cos\theta^1} = \psi \text{ (suppose)}$$

$$\psi = \theta^1 + m\sin(\theta - \theta^1) - \frac{m^2}{2}\sin 2(\theta - \theta^1) + \frac{m^3}{3}\sin 3(\theta - \theta^1) - \&c.,$$

where $m = a/a'$.

$$\text{Again } \psi = \frac{2\pi(d+a)}{M} - 2\pi\gamma, \theta^1 = \frac{2\pi d}{y^1},\ \theta = \frac{2\pi d}{y},$$

$$\therefore \frac{2\pi(d+a)}{M} - 2\pi\gamma = \frac{2\pi d}{y^1} + m\sin 2\pi d\left(\frac{1}{y} - \frac{1}{y},\right)$$

$$- \frac{m^2}{2}\sin 4\pi d\left(\frac{1}{y} - \frac{1}{y^1}\right) + \frac{m^3}{3}\sin 6\pi d\left(\frac{1}{y} - \frac{1}{y^1}\right) - \&c.$$

Hence denoting $\frac{1}{M} - \frac{1}{y^1}$ by $\frac{1}{P}$ and $\frac{1}{y} - \frac{1}{y^1} = \frac{1}{P^1}$ where P denotes the mean period of conjunction of the moon and the superior planet and P^1 the synodic period of the superior planet, we get

$$d + \frac{aP}{M} = \gamma P + \frac{P}{2\pi}\left\{ m\sin\frac{2\pi d}{P^1} - \frac{m^2}{2}\sin\frac{4\pi d}{P^1} + \frac{m^3}{3}\sin\frac{6\pi d}{P^1} - \&c. \right\}. \quad (A)$$

It is evident that all the periodic terms in [redacted] mutiple of P^1. By applying Lagrarnge' [redacted] power of m,

$$d = \gamma P - k + A\sin\frac{2\pi(\gamma P - k)}{P^1} + \text{[redacted]}$$

[redacted]

where A, B, C are determinate and $k=\frac{aP}{M}$ . ... (B)

Again

$$d_1=(\gamma+1)P-k+A\sin\frac{2\pi\{(\gamma+1)P-k\}}{P^1}-B\sin\frac{4\pi\{(\gamma+1)P-k\}}{P^1}$$

$$+C\sin\frac{6\pi\{(\gamma+1)P-k\}}{P^1}. \quad \ldots \text{(C)}$$

$$\therefore d_1-d=P+A_1\cos\frac{\pi\{(2\gamma+1)P-2k\}}{P}+B_1\cos\frac{2\pi\{(2\gamma+1)P-2k\}}{P^1}$$

$$+C_1\cos\frac{3\pi\{(1\gamma+1)P-2k\}}{P^1}, \quad \ldots \text{(D)}$$

where A_1, B_1, and C_1 are determined from (B) and (C). The equation (D) represents the length of the γth period of conjunction of the superior planet with the moon. Now let us suppose that this length of period is repeated in the γ'th period, then

$$\frac{\pi\{(2\gamma+1)P-2k\}}{P^1}+2q\pi=\frac{\pi\{(2\gamma^1+1)P-2k\}}{P^1}$$

$$\therefore \frac{P^1}{P}=\frac{\gamma^1-\gamma}{q}. \quad \ldots \text{(E)}$$

In (E), $\gamma^1-\gamma$ denotes the number of periods of conjunction observed between the $(\gamma+1)$th and (γ^1+1) th conjunction from the time whence motion is being considered. It is evident from the ... and (D) that the sum of the periodic terms in (D) in the ...nction will be zero if the equation (E) is satis- ...9·933 days and P about 28·453 days; the

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I used the convergent $\frac{137}{5}$, for Mars, *i.e.*, took 137 periods of conjunction with the moon=5 synodic periods of Mars and I found the sidereal period of Mars to be 68,762 days nearly. The first conjunction of Mars and the moon was taken on Jan. 8. 10h. 7m., G. M. T. in the year 1908 and the 138th cojunction consequently fell on September 10. 8hrs. 45min. G. M. T. in 1918. These figures were taken from the British nautical almanacs for those years. The number of days elapsed in these 137 periods was 3,897·94,305 days. Hence the motion of the moon was 142·6,685,975 revolutions, the mean sidereal month being taken at 27·321,661 days. On deducting 137, the number of periods of conjunction with the moon, Mars's motion was 5·6,605,975 revolutions in 3,897.94,305 days, whence the deduced sidereal period of Mars is 687·62 days nearly; the actual period is 686·9,797 days and according to Aryabharta it consists of 686·99,974 days nearly. It appears therefore that in trying to find when the length of the period repeated itself, he practically used a nearer convergent to $\frac{P^1}{P}$

6. Conclusion.

If we treat the moon's periods of conjunction with an inferior planet in the same way, the equation corresponding to §5 (A) becomes of the form

$$d=\gamma M'-K'+\frac{M'}{2\pi}\left\{m'\sin\frac{2\pi d}{P''}-\frac{m'^2}{2}\sin\frac{4\pi d}{P''}+\frac{m'^3}{3}\sin\frac{6\pi d}{P''}-\&c.\right\}$$

where M′ denotes the synodic month, P″ the synodic period of the inferior planet and m' in the ratio of the distance of the inferior planet and the earth from the sun. In this case the mean period becomes equal to the synodic month, and the time of revolution of an inferior planet round the earth becomes the same as that of the sun, *i.e.*, becomes equal to the sidereal year as stated by Aryabhata. In the light which is thrown by this passage from the Aryabhatiyam, it becomes clear that he is the real constructor of Indian Astronomy and not a borrower from any foreign system of Astronomy. The records of observation that he used were most probably of the previous observers of the Pataliputra school of Indian Astronomers. In the next stanza he says that "the spotless jewel of true knowledge which lay so long

sunk in the ocean of knowledge full of truth and error, has been raised by me therefrom by using the boat of my own intelligence, by the grace of God whom I worship." It seems that the position of Aryabhata in Indian Astronomy is the same as that of Ptolemy in Greek Astronomy.

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Note on Spherical Waves of Finite Amplitude

By

Sudhansukumar Banerji.

Professor Horace Lamb has pointed out to me in connection with the above problem that the notation $f(\rho')$, where $\rho'=\log \rho r^2$, ρ denoting the density is inconsistent with the equation

$$\{f'(\rho')\}^2 \frac{\partial \rho'}{\partial r}=\frac{1}{\rho}\frac{\partial p}{\partial r}.$$

[*See Bull. Cal. Math. Society*, Vol. XI, p. 85.]

It is intended to prove here that a more general assumption regarding the functional character of f leads to the same expressions for the velocities of propagation of spherical waves of finite amplitude obtained in a paper under the above title published in the *Bulletin of the Calcutta Mathematical Society*, Vol. XI.

To prove this, we start with the equations of motion and continuity,

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial r}=-\frac{1}{\rho}\frac{\partial p}{\partial r},$$

$$\frac{\partial \rho_1}{\partial t}+u\frac{\partial \rho_1}{\partial r}=-\frac{1}{r^2}\frac{\partial (ur^2)}{\partial r},$$

where $\rho_1=\log \rho$, and put

$$P=f(r,\rho_1)+u, \qquad Q=f(r,\rho_1)-u$$

For regarding r, ρ_1 as two independent variables instead of r, t, we have

$$\frac{\partial P}{\partial t}=\frac{\partial f(r,\rho_1)}{\partial \rho_1}\frac{\partial \rho_1}{\partial t}+\frac{\partial u}{\partial t},$$

$$\frac{\partial P}{\partial r}=\frac{\partial f(r,\rho_1)}{\partial r}+\frac{\partial f(r,\rho_1)}{\partial \rho_1}\cdot\frac{\partial \rho_1}{\partial r}+\frac{\partial u}{\partial r},$$

Therefore

$$\frac{\partial P}{\partial t}+u\frac{\partial P}{\partial r}=\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial r}\right)+\frac{\partial f(r,\rho_1)}{\partial \rho_1}\left(\frac{\partial \rho_1}{\partial t}+u\frac{\partial \rho_1}{\partial r}\right)$$

$$+u\frac{\partial f(r,\rho_1)}{\partial r}$$

$$=-\frac{1}{\rho}\frac{\partial p}{\partial r}-\frac{\partial f(r,\rho_1)}{\partial \rho_1}\cdot\frac{1}{r^2}\frac{\partial (ur^2)}{\partial r}+u\frac{\partial f(r,\rho_1)}{\partial r}.$$

$$=-\frac{1}{\rho}\frac{\partial p}{\partial r}-\left[\frac{\partial f(r,\rho_1)}{\partial \rho_1}\right]\frac{\partial u}{\partial r}-\frac{2u}{r}\cdot\frac{\partial f(r,\rho_1)}{\partial \rho_1}+u\frac{\partial f(r,\rho_1)}{\partial r}$$

Hence if we assume

$$\frac{2}{r}\frac{\partial f(r,\rho_1)}{\partial \rho_1}=\frac{\partial f(r,\rho_1)}{\partial r} \qquad \dots \quad (1)$$

and

$$\frac{1}{\rho}\frac{\partial p}{\partial r}=\left[\frac{\partial f(r,\rho_1)}{\partial \rho_1}\right]^2\frac{\partial \rho_1}{\partial r}+\frac{\partial f(r,\rho_1)}{\partial \rho_1}\cdot\frac{\partial f(r,\rho_1)}{\partial r}, \qquad \dots \quad (2)$$

we get

$$\frac{\partial P}{\partial t}+u\frac{\partial P}{\partial r}=-\frac{\partial f(r,\rho_1)}{\partial \rho_1}\cdot\frac{\partial P}{\partial r} \qquad \dots \quad (3)$$

and from (1) and (2)

$$\frac{1}{\rho}\frac{\partial p}{\partial r}=\left[\frac{\partial f(r,\rho_1)}{\partial \rho_1}\right]^2\left(\frac{\partial \rho_1}{\partial r}+\frac{2}{r}\right)$$

$$=\left[\frac{\partial f(r,\rho_1)}{\partial \rho_1}\right]^2\frac{\partial \log \rho r^2}{\partial r}$$

that is

$$\frac{\partial f(r,\rho_1)}{\partial \rho_1}=\left[\frac{1}{\rho}\frac{dp}{d\log \rho r^2}\right]^{\frac{1}{2}}$$

Hence from (1)

$$\frac{\partial f(r,\rho_1)}{\partial r}=\frac{2}{r}\left[\frac{1}{\rho}\frac{dp}{d\log\rho r^2}\right]^{\frac{1}{2}}.$$

Therefore

$$\frac{\partial f(r,\rho_1)}{\partial \rho_1}d\rho_1+\frac{\partial f(r,\rho_1)}{\partial r}dr=\left[\frac{1}{\rho}\frac{dp}{d\log\rho r^2}\right]^{\frac{1}{2}}d\rho_1$$

$$+\frac{2}{r}\left[\frac{1}{\rho}\frac{dp}{d\log\rho r^2}\right]^{\frac{1}{2}}dr.$$

This gives

$$f(r,\rho_1)=\int\left\{\left[\frac{1}{\rho}\frac{dp}{d\log\rho r^2}\right]^{\frac{1}{2}}\frac{d\rho}{\rho}+\left[\frac{1}{\rho}\frac{dp}{d\log\rho r^2}\right]^{\frac{1}{2}}\frac{2dr}{r}\right\}\dots \quad (4)$$

From (3) we get

$$dP=\left[dr-\left\{\frac{\partial f(r,\rho_1)}{\partial \rho_1}+u\right\}dt\right]\frac{\partial P}{\partial r},$$

Hence $dP=0$ or P is constant for a geometrical point moving with the velocity

$$\frac{dr}{dt}=\left[\frac{dp}{d\log\rho r^2}\right]^{\frac{1}{2}}+u$$

Similarly,

$$\frac{\partial Q}{\partial t}=\frac{\partial f(r,\rho_1)}{\partial \rho_1}\cdot\frac{\partial \rho_1}{\partial t}-\frac{\partial u}{\partial t}$$

$$\frac{\partial Q}{\partial r}=\frac{\partial f(r,\rho_1)}{\partial r}+\frac{\partial f(r,\rho_1)}{\partial \rho_1}\frac{\partial \rho_1}{\partial r}-\frac{\partial u}{\partial r}.$$

Therefore

$$\frac{\partial Q}{\partial t}+u\frac{\partial Q}{\partial r}=-\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial r}\right)+\frac{\partial f(r,\rho_1)}{\partial \rho_1}\left(\frac{\partial \rho_1}{\partial t}+u\frac{\partial \rho_1}{\partial r}\right)$$

$$+u\frac{\partial f(r,\rho_1)}{\partial r}$$

$$=\frac{1}{\rho}\frac{\partial p}{\partial r}-\frac{\partial f(r,\rho_1)}{\partial \rho_1}\cdot\frac{1}{r^2}\frac{\partial (ur^2)}{\partial r}+u\frac{\partial f(r,\rho_1)}{\partial r}$$

$$=\frac{1}{\rho}\frac{\partial p}{\partial r}-\frac{\partial f(r,\rho_1)}{\partial \rho_1}\cdot\frac{\partial u}{\partial r}-\frac{2u}{r}\frac{\partial f(r,\rho_1)}{\partial \rho_1}+u\frac{\partial f(r,\rho_1)}{\partial r}$$

which becomes by virtue of (1) and (2)

$$\frac{\partial Q}{\partial t}+u\frac{\partial Q}{\partial r}=\frac{\partial f(r,\rho_1)}{\partial \rho_1}\cdot\frac{\partial Q}{\partial r}.$$

Therefore

$$dQ=\left[dr-\left\{-\left(\frac{1}{\rho}\frac{dp}{d\log\rho r^2}\right)^{\frac{1}{2}}+u\right\}dt\right]\frac{\partial Q}{\partial r}.$$

Therefore $dQ=0$, or Q is constant for a geometrical point moving with the velocity

$$\frac{dr}{dt}=-\left[\frac{1}{\rho}\frac{dp}{d\log\rho r^2}\right]^{\frac{1}{2}}+u.$$

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Fundamental Relations in Homogeneous Co-ordinates.

By

Prof. C. V. H. Rao,

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Two relations are mentioned in all the books on the subject, namely

$$ax+by+cz=2\Delta, \qquad \dots \quad (1)$$

and

$$\Sigma a(p-q)(p-r)=4\Delta^2, \qquad \dots \quad (2)$$

where a, b, c, Δ have the usual significance with reference to the fundamental triangle, and p, q, r are the distances of the vertices from an arbitrary line.

It is the object of this note to point out that these relations are not unconnected. For this purpose it is useful to replace relation (2) by

$$apx+bqy+crz=d, \qquad \dots \quad (3)$$

where x, y, z, d are the distances of an arbitrary point from the sides of the triangle and the arbitrary line; on replacing d by the usual expression for it we obtain relation (2).

The relation (3) may well be called a fundamental relation for it has a dual interpretation. First considering x, y, z, d as variables it is the linear relation connecting the distances of an arbitrary point from four given lines, and secondly taking p, q, r, d as variables it is the linear relation connecting the distances of four fixed points from an arbitrary line. And in both these respects the relation is capable of being exhibited more symmetrically.

The connection between the relations (1) and (3) is that on reciprocating the former with respect to circle having an arbitrary point for centre we obtain the latter. The proof of this statement is quite straightforward and depends on noting how the distance of a point from a line and the distance of two points are altered on reciprocation.

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ON THE ϕ CONIC OF TWO CONICS.

BY

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University Professor, Lahore.

The equation to the ϕ conic is usually obtained by straightforward analysis—not very elegant—or by appealing to the idea of Invariance, which is unfair since it only shifts the burden. The object of this note is to direct attention to a different method which is largely geometrical, and affords a simple example of the usefulness of thinking of a symbolical expression as a symbol.

We denote a point P by the single symbol P, which may be thought of as representing the left-hand side of its tangential equation, or as the conjoint of x, y, or simply as a symbol serving to name and to identify the point at once.[1] Any point collinear with P, P′ is then denoted by $\lambda P+P'$ where λ is a number. The four elements P, P′, $\pm\lambda P+P'$, form a harmonic range; the elements may be points or lines, circle of a coaxal system, or conics through four points or indeed any two curves of the same order. $\lambda P+P'$ represents in fact what is a "rete" with Italian writers.

The four conics S, S′, $\pm\lambda S+S'$ are then harmonic, their tangents at any one of the common points are harmonic; in particular if they are circles their centres are harmonic. Now it is a result of elementary geometry and easy to verify that if P, P′ and Q, Q′ are two harmonic pairs of an involution on a line, then the middle points of PP′ and QQ′ are also a pair of the involution and therefore harmonic with the double points.

Thus if a line meet two circles harmonically, the two circles of the coaxal system which it touches are necessarily of the form $\pm\lambda S+S'$; and the result holds if S, S′ represent two conics.

[1] For a more complete exposition compare Baker, *Proc. Camb. Phil. Soc.*, Vol. XX, p. 140, 1920.

Any arbitrary line l, m, n touches two conics of the system $\lambda S+S'$ whose parameters are given by

$$\lambda^2\Sigma+2\lambda\phi+\Sigma'=0,$$

where Σ, Σ' are the tangential equations of S, S'. In virtue of what was proved above it follows that any line which cuts S, S' harmonically touches two conics of the system which are necessarily of the form $\pm\lambda S+S'$. The envelope of such lines is therefore $\phi=0$. The F conic permits of precisely an analogous treatment.

NOTICES RESPECTING NEW BOOKS PRESENTED TO THE LIBRARY

The undermentioned books have been presented to the library for which the Society tenders its best thanks:—

1. Modern methods for measuring the intensity of gravity. By Clarence H. Swick. Pp. 1—96. U. S. Coast and Geodetic Survey, 1921.

2. Elements of Map projection with application to map and chart construction. By Charles H. Deetz and Oscar S. Adams. Pp. 1—163 and eight plates. U. S. Coast and Geodetic Survey, 1921.

3. Relations between plane rectangular coordinates and geographic positions. By Walter F. Reynolds. Pp. 1—90. U. S. Coast and Geodetic Survey, 1921.

4. Apercus théoriques sur la Résistance des Fluides. Par Henri Villat. Pp. 1—101. Gauthier-Villars et Cie, 1920.

5. Essai Philosophique sur les Probabilitiés. Par P. S. Laplace. Tome I, pp. 1—101. Tome II, pp. 1—108. Gauthier-Villars et Cie, Paris, 1921.

6. Memoires sur L'E'lectromagnétisme et L'E'lectrodynamique. Par A. M. Ampère. Pp. 1—110. Gauthier-Villars et Cie, Paris, 1921.

7. Reprints of Papers from the Science Laboratories of the University of Sydney, 1916-17 to 1919-20.

8. Triangulation in Rhode Island. By Earl Church. Pp. 1—97 and eight figures. U. S. Coast and Geodetic Survey, 1920.

9. Latitude developments connected with Geodesy and Cartography with tables including a table for Lambert equal area meridional projection. By O. S. Adams. Pp. 1—132. U. S. Coast and Geodetic Survey, 1921.

10. Annual Report of the Superintendent of the United States Coast and Geodetic Survey, 1918.

11. Annual Report of the Smithsonian Institution, 1918.

12. General Theory of the Lambert Conformal Conic Projection. By O. S. Adams. Pp. 1—37. U. S. Coast and Geodetic Survey, 1918.

13. Description of the U. S. Coast and Geodetic Survey Tide-Predicting Machine, No. 2, 1915.

14. Annual Report of the Director, United States Coast and Geodetic Survey, 1920.

15. Investigations of gravity and Isostasy. By William Bowie. Pp. 1—196 with eighteen illustrations. U. S. Coast and Geodetic Survey, 1917.

16. Wire-drag Work on the Atlantic Coast. By N. H. Heck and S. H. Sawley. Pp. 1—24. U. S. Coast and Geodetic Survey, 1915.

17. Latitude Observations with Photographic Zenith Tube at Gaithersburg, MD. By Frank E. Ross. Pp. 1—127 with seventeen illustrations. U. S. Coast and Geodetic Survey, 1915.

18. Lambert projection tables for the United States. By O. S. Adams. Pp. 1—243. U. S. Coast and Geodetic Survey, 1918.

19. Elements of Chart Making. By E. L. Jones. U. S. Coast and Geodetic Survey, 1916.

20. Use of mean sea level as the datum for elevations. By E. L. Jones. Pp. 1—21. U. S. Coast and Geodetic Survey, 1921.

21. Application of the Theory of Least Squares to the adjustment of triangulation. By O. S. Adams. Pp. 1—220. U. S. Coast and Geodetic Survey, 1915.

22. The Lambert Conformal Conic Projection with two standard parallels including a comparison of the Lambert Projection with the Bonne and Polyconic Projections. By C. H. Deetz. Pp. 1—61 with seven illustrations. U. S. Coast and Geodetic Survey, 1918.

23. Fourth general adjustment of the precise level net in the United States and the Resulting Standard Elevations. By William and H. G. Avers. Pp. 1—328 and five illustrations. U. S. Coast and Geodetic Survey, 1914.

24. Lambert projection tables with conversion tables supplement to the Lambert Conformal Conic Projection with two standard parallels. By C. H. Deetz. Pp. 1—84. U. S. Coast and Geodetic Survey, 1918.

25. Experiments in Aeroplane Photo Surveying. By Major C. G. Lewis and Capt. H. G. Salmond. Professional Paper, No. 19. Survey of India, Dehra-Dun, 1920.

26. A study of map projections in general. By O. S. Adams. U. S. Coast and Geodetic Survey, 1919.

Notes and News

The following letter addressed to the Secretary from Professor G. Mittag-Leffler will be read with interest by the members of the Society:—"Dear Sir, I beg you please to accept yourself and to convey to your honoured Society my best thanks for the hearty felicitations on account of my 75th birthday on March 16th last, contained in your letter of October 11th. It belongs to my dreams of future—very uncertain, however—once to go and see your beautiful and interesting country, which has always tempted me, and then to make the personal acquaintance of your mathematicians. If you want it, I should take pleasure in sending you a little article for publication in your Proceedings." The Society affords the venerable Professor a hearty welcome and very much desires that his dreams may fructify at an early date.

* * *

The following notes have been sent to us by Professor G. H. Bryan with regard to his paper on graphic solutions of spherical triangles published in the September issue of the Bulletin (Vol. XII, No. 2):—

"Since sending the MS. to you I have seen a paper on the same subject by Bradley in the American Mathematical Monthly. His constructions, however, are greatly inferior to mine in the following respects:

(1) He does not show how to construct a model of the trihedral angles represented by the parts of the spherical triangle whereas my figures when cut out and folded automatically join up and form the model in question.

(2) His figures are rendered unnecessarily complicated by construction lines which are not necessary according to my methods. In my figures the only parts which are purely constructional are the arcs of circles used in measuring off equal segments on different lines. These are not really necessary but are rather introduced for the sake of greater clearness of exposition, and they serve an additional useful function in showing which parts must be joined together in making up the model.

(3) In the notation adopted in my paper parts of the figure which are common to two faces of the model are represented by the same letters in each with different suffixes thus indicating at a glance how

the model is to be joined up. Bradley's notation on the other hand is far from satisfactory."

* * * *

The following letter by Professor G. A. Miller, published in *Science*, Vol. LIV, No. 1396, September 30, 1921, on a new definition of pure mathematics is a matter for discussion and is one on which every mathematician may have his own say:—

During the present year there appeared a volume of the *Acta Mathematica*, volume 38, which was dedicated to the memory of H. Poincaré, the noted French mathematician who died in 1912. This volume opens with an account of his own works by Poincaré in which he deals briefly with his own contributions to the advancement of various subjects. This is followed by a report on the theory of groups and the works of E. Cartan, which Poincaré read before the council of faculty of sciences of the University of Paris on the eve of the operation resulting in his death. The rest of the volume is devoted to letters and to various articles written by others but relating to Poincaré and his works.

In the present note we desire to direct attention to the second article mentioned above, which seems to be one of the last articles, if not the last article, written by Poincaré, and contains some remarkable statements in regard to the theory of groups. One of these is as follows: "The theory of groups is so to say, entire mathematics, divested of its matter and reduced to a pure form." The interest in this statement should be increased by the fact that it may be regarded as a new definition of pure mathematics, the skyscraper among scientific structures. One of the best known other definitions is due to B. Peirce, who stated that "mathematics is the science which draws necessary conclusions." It should, however, not be inferred that the latter definition has been generally accepted as an entirely satisfactory one, nor do we want to create the impression that the former is likely to be universally adopted.

It may, however, be a matter of wide interest to see what Poincaré meant by the statement quoted above. Such an insight can probably be best gained by reading his own preliminary remarks, which are, in part, as follows:

The preponderant rôle of the theory of groups in mathematics has been unsuspected for a long time. Eighty years ago even the name of group was unknown. It was Galois who first had a clear notion of it, but it is only since the works of Klein, and especially of Lie, that one has begun to see that there is almost no mathematical theory in which

this notion does not occupy an important place. . . . It is necessary to give the same name to different things, but on condition that these things are different as to matter but not as to form. What is the cause of the mathematical phenomenon so often constant? And, on the other hand, of what consists the community of form which subsists under the diversity of matter? It is due to this that every mathematical theory is, in the last analysis, the study of properties of a group of operations, that is to say, of a system formed by certain fundamental operations and of all the combinations which can be made therefrom.

If, in another theory, one studies other operations which combine according to the same laws one will naturally see a set of theorems, having a one to one correspondence to those of the first theory, unfold themselves, and the two theories may be developed with a perfect parallelism; an artifice of language like those of which we just spoke, suffices to make this parallelism manifest and to give almost the impression of a complete identity. One says then that the two groups of operation are isomorphic, or that they have the same structure. If then one divests the mathematical theory of this which appertains to it only by accident, that is to say, its matter, there will remain only the essential, that is to say, the form; and this form, which constitutes, so to say, the solid skeleton of the theory, will be the structure of the group.

* * *

In the *Messenger of Mathematics*, Vol. LI, May, 1921, Dr. J. W. L. Glaisher writes on the early history of the signs of + and — and on the early German arithmeticians. "It was at one time believed that the signs + and — were introduced into Algebra by Stifel in his *Arithmetica Integra* of 1544, but in 1864 DeMorgan contributed to the Cambridge Philosophical Society a paper in which he showed that they had been used by Widman in his *Rechenung* of 1489. This fact had, however, been previously noticed and pointed out by Drobisch in 1840; and his discovery of the signs had been noted by Gerhardt in 1843 and by Cantor in 1857. DeMorgan in his paper not only drew attention to the existence of the signs + and — in Widman's book but he inferred from the mode of their occurrence that Widman or some predecessor had derived them 'from the warehouse,' so that they had a commercial and not an arithmetical origin." Dr. Glaisher finds himself unable to agree with most of DeMorgan's conclusions and suggestions and considers that with respect to the origins of the signs + and —, all the evidence shows that they were derived from algebra and not from commerce. These evidence form the subject matter of the present paper.

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On the Disturbed Electron orbits in an Electro-magnetic field.

By

Panchanan Das, M.Sc.

[*Read, September 26th, 1921.*]

The Quantum-theory of spectra, as it now stands, is incomplete in so far as we cannot explain, with its help the class of phenomena like absorption, dispersion, resonance-radiation etc. The difficulty is that we do not know what happens when an external electric or magnetic field acts on a quantitised (gequantelte) system. Ehrenfest and Burgers' theory of adiabatic invariants marks the first step towards a solution of this problem. It may be ennunciated as follows :—The quantum-path of a system without any external field adjusts itself to the new quantum-path with field in a mechanical manner provided that the external field grows slowly.* It will appear presently that when a rapidly alternating electro-magnetic field acts on such a "quantitised" system, it is sometimes possible to regard its readjustment to the external field as a reversible one, and hence the principle of "mechanical transformability" may be extended to such systems. We shall thus investigate the effect of an electro-magnetic field on a typical atom consisting of a nucleus and the valency electron. The method followed will be slightly different from that of Debye in his well known work on the dispersion of light by the hydrogen-molecule. We shall thus see that the mechanism of selective absorption and resonance-radiation are better understood in the light of these calculations.

We assume, for simplicity, that the valency electron is uninfluenced by the remaining electrons. We also suppose that the external electro-magnetic field consists of a planepolarised wave. Let the coordinates of the valency electron referred to the nucleus as origin by x, y, z. We choose our axes in such a manner that the resultant electric and magnetic forces are parallel to the X— and Z— axes respectively, and

* "Die Quantenbahn ohne Feld wird bei allmahlichen Einschalten des Feldes auf mechanischem Wege in die Quantenbahn "ubergefuhrt."—Atombau und Spektrallinien-Sommerfeld

the direction of propagation is parallel to the Y axis Then the components of the field are the real parts of :—

$$\left.\begin{aligned} &E_x = E\cdot\epsilon^{\frac{2\pi i}{\lambda}(vt-y)} \quad E_y=0,\ E_z. \\ &H_x=0,\ H_y=0,\ H_z=H\cdot\epsilon^{\frac{2\pi i}{\lambda}(vt-y)}. \end{aligned}\right\} \quad \ldots \quad (1)$$

If the central charge be Ze, the equations of motion are :—

$$m\ddot{x} = -\frac{Ze^2}{r^2}\cdot\frac{x}{r} + e\cdot E_x + \frac{e}{c}\cdot H_z\dot{y}$$

$$m\ddot{y} = -\frac{Ze^2}{r^2}\cdot\frac{y}{r} - \frac{e}{c}\cdot H_z\cdot\dot{x}$$

$$m\ddot{z} = -\frac{Ze^2}{r^3}\cdot\frac{z}{r}.$$

If we neglect $\dot{x}$, $\dot{y}$ as small compared to c, these reduce to

$$m\ddot{x} = -\frac{Ze^2}{r^2}\cdot\frac{x}{r}e\cdot E_x.$$

$$m\ddot{y} = -\frac{Ze^2}{r^2}\cdot\frac{y}{r}.$$

$$m\ddot{z} = -\frac{Ze^2}{r^2}\cdot\frac{z}{r}.$$

Let A be the perihelion of the electron-orbit, P the electron, and N the nodal point of intersection of the orbit with the $x=y$ plane.

Let $\bar{\omega} = \widehat{NP}$, ω = true anomaly $= \widehat{PA}$,

i = inclination of orbit and Ω = longitude of N measured from ox. Then we have,

$$x = r\cos(\bar{\omega}+\omega)\cos\Omega + r\sin(\bar{\omega}+\omega)\cos i\sin\Omega,$$

$$y = r\cos(\bar{\omega}+\omega)\sin\Omega + r\sin(\bar{\omega}+\omega)\cos i\sin\Omega,$$

$$z = r\sin(\bar{\omega}+\omega)\sin i.$$

As a first approximation we may limit ourselves to the circular orbit, and thus put $r=a$, and $\omega=nt=$mean anomaly.

It is thus obvious from the equations of motion that

$$n^2=\frac{Ze^2}{ma^3}, \text{ or } Z=\frac{mn^2a^3}{e^2}.$$

The equations of motion can be solved only approximately, and even that in special cases. So we restrict ourselves to a few special cases only.

Case I. Let $\Omega=0$, and $i=90°$ for all values of time. We suppose that the disturbance takes place in the coordinate r and $\bar{\omega}$ only. Thus,

$$x=r\cos(\bar{\omega}+nt),\ y=0,\ z=r\sin(\bar{\omega}+nt).$$

The equations of motion are

$$m\ddot{x}=-\frac{mn^2a^3}{r^2}\cdot\frac{x}{r}+e\cdot E_x,$$

$$m\ddot{z}=-\frac{mn^2a^3}{r^2}\cdot\frac{z}{r}.$$

It is easily seen that after actual differentiation and substitution we have,

$$\ddot{z}=\frac{\ddot{r}}{r}z+\frac{2\dot{r}x}{r}(\dot{\omega}+n)+x\ddot{\omega}-z(\dot{\omega}+n)^2$$

and $$\ddot{x}=\frac{\ddot{r}}{r}x-\frac{2\dot{r}z}{r}(\dot{\omega}+n)-z\ddot{\omega}-x(\dot{\omega}+n)^2.$$

Substituting these in the equations of motion after putting $r=a$, and neglecting $\dot{\omega}^2$ and $\dot{r}\dot{\omega}$, we get,

$$\frac{\ddot{r}}{a}z+\frac{2\dot{r}xn}{a}+x\ddot{\omega}-2z\dot{\omega}n=0 \qquad \dots (2)$$

and $$\frac{\ddot{r}}{r}x-\frac{2\dot{r}zn}{a}-z\ddot{\omega}-2x\dot{\omega}n=\frac{e}{m}\cdot E\cdot\epsilon^{\frac{2\pi i}{\lambda}(vt-y)}$$

$$=\frac{e}{m}\cdot E\cdot\epsilon^{\frac{2\pi i}{\lambda}vt}, \text{ since } y=0,$$

$$=\frac{e}{m}\cdot E\cdot\cos n't, \qquad \dots (3)$$

if we put $n' = \frac{2\pi V}{\lambda}$ and retain the real part only. Multiplying (2) by x and (3) by z, and by subtruction, we get

$$2\dot{r}an + a^2\ddot{\omega} = -z\cdot\frac{e}{m}\cdot E\cdot\cos n't.$$

or $$a\ddot{\omega} + 2\dot{r}n = -\frac{e}{m}\cdot E\cdot\sin nt\cos n't. \qquad \dots \quad (4)$$

Again multiplying (2) by z and (3) by x and by addition we get

$$\ddot{r} - 2a\dot{\omega}n = \frac{e}{m}\cdot E\cos nt\cdot\cos n't. \qquad \dots \quad (5)$$

Eliminating $\dot{\omega}$ between (4) and (5), we get

$$\dddot{r} + 4n^2\dot{r} = -2n\cdot\frac{e}{m}\cdot E\sin nt\cos n't + \frac{e}{m}E\cdot\frac{d}{dt}(\cos nt\cos n't).$$

The solution is

$$\dot{r} = A\sin(2nt+\alpha) + \frac{1}{D^2+4n^2}\Bigg[n\cdot\frac{e}{m}\cdot E\left\{\frac{\cos(n+n')t}{n+n}\right.$$

$$\left.+\frac{\cos(n-n')t}{n-n'}\right\} + \frac{e}{2m}E\left\{\cos(n+n')t + \cos(n-n')t\right\}\Bigg],$$

where $$D \equiv \frac{d}{dt}.$$

We can also solve for $\breve{\omega}$ in the same manner.

Case II. Let $\Omega=0$, $\iota=0$, for all values of time. The equations of motion here reduce to

$$\ddot{r}a - 2a^2\dot{\omega}n = \frac{e}{m}\cdot E\cdot x\cdot\epsilon^{\frac{2\pi\iota}{\lambda}(Vt-y)}$$

$$2\dot{r}\cdot an + a^2\ddot{\omega} = -\frac{e}{m}E\cdot y\cdot\epsilon^{\frac{2\pi\iota}{\lambda}(Vt-y)}.$$

Also $$y = a\sin(\omega + nt).$$

If we regard y as small compared with Vt, we can expand $\epsilon^{\frac{2\pi\iota}{\lambda}(Vt-y)}$ in ascending powers of y and retain the first few terms. Thus

$$\epsilon^{\frac{2\pi\iota}{\lambda}(Vt-y)} = (\cos n't - \iota \sin n't)\left(1 + \frac{2\pi\iota}{\lambda} y + \ldots\right)$$

It is obvious that since y is of the form $a \sin nt$, (neglecting $\bar{\omega}$), the real part of the above can be expanded in a series of the form $\sum_{pq} A_{p,q} \cos(pn+qn')t$.

It thus appears that the general solution of the problem would be of the form $\sum_{pq} B_{p,q} \cos(pn+qn')t$.

We proceed to examine the particular form $\cos(n-n')t$ only. If the external electro-magnetic wave is nearly of the same frequency as the orbital motion of the electron, the difference $n-n'$ is small compared with either n or n'. Hence the effect of the field is to cause a slow periodic disturbance in the elements of the orbit. The periods of the disturbance and the orbital motion are $\frac{2\pi}{n-n'}$, and $\frac{2\pi}{n}$ respectively. Let T be the average period between two successive collisions of the molecules. Then we may suppose, that T is large compared with $\frac{2\pi}{n}$ and $\frac{2\pi}{n-n'}$ is large compared with T.

i.e.
$$\frac{2\pi}{n} < T < \frac{2\pi}{n-n'}.$$

It is reasonable to suppose that as soon as a collision takes place, the disturbance is reversed or arrested or at least modified. If we limit ourselves to small intervals of time $\delta t < T$, then the periodic character of the disturbance is not manifest; the disturbance should rather increase or decrease steadily with time δt.

Thus if the total energy of the atomic system without the field be W then after an interval δt measured from the instant the field is introduced its total energy is $W + \frac{dW}{dt}\delta t$ where $\frac{dW}{dt}$ otherwise containing a periodic factor, is fairly constant within the interval δt. If a collision now takes place, we may expect a quantum-radiation, and the frequency of it will be $\nu + \frac{d\nu}{dt}\delta t$. Thus instead of a sharply defined

line we shall now have a diffuse line. If the temperature rises, the frequency of collisions will also rise, and hence T will be smaller, and the diffuseness measured by $\frac{dv}{dt}\delta t$ will also fall off. Hence with a rise of temperature the lines should become more sharply defined. In this connection we may refer to an experiment by Wood [Physic Rev. XI, 1918], in which he allows the sodium light D_1 to be incident on a quartz tube containing sodium vapour, and examines the resonance radiation. He finds that the resonance-lines are at first very diffuse and ill-defined, but as the temperature rises, the lines become sharper and more clearly defined.

Bohr's correspondence-principle makes the phenomenon of resonance radiation very intelligible after these calculations. As in Wood's experiment, suppose that a mass of sodium vapour is excited by the D_1 light. Now in this vapour there is present a large number of sodium atoms in a state to emit the D_2 radiation, and as the orbital frequency of the external field, a disturbance of the orbit would ensue, and this circumstance may be supposed to assist the possible radiation, *viz.*, the D_2 radiation and we should expect a maximum in the position of the D_2 line. This result however is very general.

My thanks are due to Professor C. V. Raman for the helpful interest he has taken in this paper, and also to Dr. D. M. Bose for his useful criticism.

THE OSCULATING CONIC AT INFINITY

BY

GURUDAS BHAR.

The object of the present paper is to investigate the equation of the osculating conic to an algebraic curve when the point of contact is at infinity. About the nature of an algebraic curve at infinity, mathematicians have apparently not proceeded much further in their investigations than the determination of the asymptote when the point at infinity is not a singular point and of the nature of singularity when it is a singular point. The determination of the osculating conic at infinity, which is not a difficult task, gives a fuller insight into the form of the curve at infinity when the point at infinity is not a singular point. In connection with the investigations of Prof. S. Mukhopadhyaya* about the way in which the osculating conic varies as the point of contact is moved along the curve, the determination of the limiting form of the osculating conic when the point of contact goes to infinity is of special importance. This paper has been written at the suggestion and under the guidance of Prof. S. Mukhopadhyaya.

1. Let the equation to a plane algebraic curve of the nth order be

$$u_n+u_{n-1}+u_{n-2}+\dots+u_2+u_1+u_0=0$$

where u_r denotes the homogeneous terms of the rth degree in x and y This can also be written in the following equivalent form

$$x^n f_n\left(\frac{y}{x}\right)+x^{n-1}f_{n-1}\left(\frac{y}{x}\right)+x^{n-2}f_{n-2}\left(\frac{y}{x}\right)+\dots=0. \qquad \dots \quad (1)$$

Consider the curve

$$y=ax+\beta+\frac{\gamma}{x}+\frac{\delta}{x^2}+\frac{\epsilon}{x^3}+\frac{\phi}{x^4} \qquad \dots \quad (2)$$

* *Vide*—New methods in the Geometry of a Plane Arc · *Bulletin C.M.S. Vol. I, No. I, 1909.*

which meets the curve (1) at points of which the abscissae are given by

$$x^n f_n\left(\alpha+\frac{\beta}{x}+\frac{\gamma}{x^2}+\frac{\delta}{x^3}+\frac{\epsilon}{x^4}+\frac{\phi}{x^5}\right)$$

$$+x^{n-1} f_{n-1}\left(\alpha+\frac{\beta}{x}+\frac{\gamma}{x^2}+\frac{\delta}{x^3}+\frac{\epsilon}{x^4}+\frac{\phi}{x^5}\right)$$

$$+x^{n-2} f_{n-2}\left(\alpha+\frac{\beta}{x}+\frac{\gamma}{x^2}+\frac{\delta}{x^3}+\frac{\epsilon}{x^4}+\frac{\phi}{x^5}\right)+\ldots=0$$

that is by

$$x^n f_n(\alpha)+x^{n-1}\{\beta f'_n(\alpha)+f_{n-1}(\alpha)\}$$

$$+x^{n-2}\{\gamma f'_n(\alpha)+\frac{\beta^2}{2} f''_n(\alpha)+\beta f'_{n-1}(\alpha)+f_{n-2}(\alpha)\}$$

$$+x^{n-3}\{\delta f'_n(\alpha)+\beta\gamma f''_n(\alpha)+\frac{\beta^3}{6} f'''_n(\alpha)+\gamma f'_{n-1}(\alpha)$$

$$+\frac{\beta^2}{2} f''_{n-1}(\alpha)+\beta f'_{n-2}(\alpha)+f_{n-3}(\alpha)\}$$

$$+x^{n-4}\{\epsilon f'_n(\alpha)+\frac{2\beta\delta+\gamma^2}{2} f''_n(\alpha)+\frac{\beta^2\gamma}{2} f'''_n(\alpha)+\frac{\beta^4}{24} f^{iv}_n(\alpha)$$

$$+\delta f'_{n-1}(\alpha)+\beta\gamma f''_{n-1}(\alpha)+\frac{\beta^3}{3!} f'''_{n-1}(\alpha)+\gamma f'_{n-2}(\alpha)$$

$$+\frac{\beta^2}{2!} f''_{n-2}(\alpha)+\beta f'_{n-3}(\alpha)+f_{n-4}(\alpha)\}$$

$$+x^{n-5}\{\phi f'_n(\alpha)+(\gamma\delta+\beta\epsilon) f''_n(\alpha)+\frac{1}{2}(\beta^2\delta+\gamma^2\beta) f'''_n(\alpha)$$

$$+\frac{1}{6}\beta^3\gamma f^{iv}_n(\alpha)+\frac{\beta^5}{5!} f^{v}_n(\alpha)+\epsilon f'_{n-1}(\alpha)+\left(\frac{\gamma^2}{2}+\beta\delta\right) f''_{n-1}(\alpha)$$

$$+\frac{1}{2}\beta^2\gamma f'''_{n-1}(\alpha)+\frac{\beta^4}{4!} f^{iv}_{n-1}(\alpha)+\delta f'_{n-2}(\alpha)+\beta\gamma f''_{n-2}(\alpha)+\frac{\beta^3}{6} f'''_{n-2}(\alpha)$$

$$+\gamma f'_{n-3}(\alpha)+\frac{\beta^2}{2} f''_{n-3}(\alpha)+\beta f'_{n-4}(\alpha)+f_{n-5}(\alpha)\}+\ldots$$

Five consecutive points of intersection will be at infinity if the co-efficients of $x^n, x^{n-1}, x^{n-2}, x^{n-3}, x^{n-4}$ all vanish ; *i.e.*, if

$$\left.\begin{aligned}
&(i)\ f_n(a)=0,\\
&(ii)\ \beta f'_n(a)+f_{n-1}(a)=0,\\
&(iii)\ \gamma f'_n(a)+\frac{\beta^2}{2}f''_n(a)+\beta f'_{n-1}(a)+f_{n-2}(a)=0,\\
&(iv)\ \delta f'_n(a)+\beta\gamma f''_n(a)+\frac{\beta^3}{6}f'''_n(a)+\gamma f'_{n-1}(a)\\
&\qquad\qquad+\frac{\beta^2}{2}f''_{n-1}(a)+\beta f'_{n-2}(a)+f_{n-3}(a)=0,\\
&(v)\ \epsilon f'_n(a)+\delta\{\beta f''_n(a)+f'_{n-1}(a)\}+\frac{\gamma^2}{2}f''_n(a)\\
&\qquad+\gamma\left\{\frac{\beta^2}{2}f'''_n(a)+\beta f''_{n-1}(a)+f'_{n-2}(a)\right\}+\frac{\beta^4}{24}f^{iv}_n(a)\\
&\qquad+\frac{\beta^3}{6}f'''_{n-1}(a)+\frac{\beta^2}{2}f''_{n-2}(a)+\beta f'_{n-3}(a)+f_{n-4}(a)=0.
\end{aligned}\right\}\ \ldots\ (3)$$

Let a which is supposed to be real be a non-multiple root of the equation $f_n(a)=0$; then from (*ii*) above

$$\beta=-\frac{f_{n-1}(a)}{f'_n(a)}\qquad\ldots\ (4)$$

which we shall write as $-\dfrac{f_{n-1}}{f'_n}$; similarly from (*ii*) and (*iii*)

$$\gamma=-\frac{1}{2}\,\frac{1}{f'^3_n}\,\{f^2_{n-1}f''_n-2f_{n-1}f'_{n-1}f'_n+2f_{n-2}f'^2_n\}.\qquad\ldots\ (5)$$

From (*iv*) substituting the values of β and γ we get

$$\begin{aligned}
\delta=\frac{1}{6}\,\frac{1}{f'^5_n}\{&f^3_{n-1}f'''_nf'_n-3f^3_{n-1}f''^2_n-3f^2_{n-1}f''_{n-1}f'^2_n\\
&+9f^2_{n-1}f'_{n-1}f'_nf''_n+6f_{n-1}f'_{n-2}f'^3_n-6f_{n-1}f_{n-2}f'^2_nf''_n\\
&-6f_{n-1}f'^2_{n-1}f'^2_n+6f_{n-2}f'_{n-1}f'^3_n-6f_{n-3}f'^4_n\}.
\end{aligned}\qquad\ldots\ (6)$$

Similarly using the values of β, γ and δ obtained above we deduce from (v)

$$
\begin{aligned}
\epsilon=-\frac{1}{24}\,\frac{1}{f'^7_n}\{ & f^4_{n-1}f'^2_nf^{iv}_n-10f^4_{n-1}f'_nf''_nf'''_n+15f^4_{n-1}f''^3_n \\
& +18f^3_{n-1}f''_{n-1}f'^2_nf''_n-60f^3_{n-1}f'_{n-1}f'_nf''^2_n+16f^3_{n-1}f'_{n-1}f'''_nf'^2_n \\
& -4f^3_{n-1}f'''_{n-1}f'^3_n-30f^2_{n-1}f'_{n-2}f'^3_nf''_n+72f^2_{n-1}f'^2_{n-1}f'^2_nf''_n \\
& +36f^2_{n-1}f_{n-2}f'^2_nf''^2_n-24f^2_{n-1}f'_{n-1}f''_{n-1}f'^3_n+12f^2_{n-1}f''_{n-2}f'^4_n \\
& -12f_{n-2}f^2_{n-1}f'^3_nf'''_n-48f_{n-1}f_{n-2}f'_{n-1}f'^3_nf''_n \\
& +36f_{n-1}f'_{n-1}f'_{n-2}f'^4_n-24f_{n-1}f'_{n-1}f_{n-2}f'^3_nf''_n \\
& -24f_{n-1}f'^3_{n-1}f'^3_n+12f_{n-2}f_{n-1}f''_{n-1}f'^4_n-24f_{n-1}f'_{n-2}f'^5_n \\
& +24f_{n-3}f_{n-1}f'^4_nf''_n+12f^2_{n-2}f'^4_nf''_n+24f_{n-2}f'^2_{n-1}f'^4_n \\
& -12f_{n-2}f'_{n-2}f'^5_n-24f_{n-3}f'_{n-1}f'^5_n+24f_{n-4}f'^6_n\}. \qquad \ldots \quad (7)
\end{aligned}
$$

If a, β, γ, δ and ϵ have the values determined above, the curve

$$y=ax+\beta+\frac{\gamma}{x}+\frac{\delta}{x^2}+\frac{\epsilon}{x^3}+\frac{\phi}{x^4}$$

will have five pointic contact with the curve (1) at infinity.

2. Let us, as usual, denote by p, q, r, s and t the first five differential co-efficients of y with respect to x. For the curve (2), *viz.*,

$$y=ax+\beta+\frac{\gamma}{x}+\frac{\delta}{x^2}+\frac{\epsilon}{x^3}+\frac{\phi}{x^4}$$

we have

$$
\left.
\begin{aligned}
p &= a-\frac{\gamma}{x^2}-\frac{2\delta}{x^3}-\frac{3\epsilon}{x^4}-\frac{4\phi}{x^5}, \\
q &= \frac{2\gamma}{x^3}+\frac{6\delta}{x^4}+\frac{12\epsilon}{x^5}+\frac{20\phi}{x^6}, \\
r &= -\frac{6\gamma}{x^4}-\frac{24\delta}{x^5}-\frac{60\epsilon}{x^6}-\frac{120\phi}{x^7}, \\
s &= \frac{24\gamma}{x^5}+\frac{120\delta}{x^6}+\frac{360\epsilon}{x^7}+\frac{840\phi}{x^8}, \\
t &= -\frac{120\gamma}{x^6}-\frac{720\delta}{x^7}-\frac{2520\epsilon}{x^8}-\frac{6720\phi}{x^9}.
\end{aligned}
\right\} \qquad \ldots \quad (8)
$$

Now as the curves (1) and (2) have five consecutive points at infinity common, the limiting values of p, q, r and s at infinity for the two curves (1) and (2) are the same. Hence the values of p, q, r and s at infinity for the curve (1) are given by

$$\left.\begin{aligned} p &= \underset{x=\infty}{\mathrm{Lt}} \left\{ a - \frac{\gamma}{x^2} - \frac{2\delta}{x^3} - \frac{3\epsilon}{x^4} - \frac{4\phi}{x^5} \right\}, \\ q &= \underset{x=\infty}{\mathrm{Lt}} \left\{ \frac{2\gamma}{x^3} + \frac{6\delta}{x^4} + \frac{12\epsilon}{x^5} + \frac{20\phi}{x^6} \right\}, \\ r &= \underset{x=\infty}{\mathrm{Lt}} \left\{ -\frac{6\gamma}{x^4} - \frac{24\delta}{x^5} - \frac{60\epsilon}{x^6} - \frac{120\phi}{x^7} \right\}, \\ s &= \underset{x=\infty}{\mathrm{Lt}} \left\{ \frac{24\gamma}{x^5} + \frac{120\delta}{x^6} + \frac{360\epsilon}{x^7} + \frac{840\phi}{x^8} \right\}, \end{aligned}\right\} \quad \ldots \quad (9)$$

where a, β, γ, δ and ϵ have the values already determined.

The general equation of the osculating conic as given by Prof. S. Mukhopadhyaya in his paper on "A general Theory of Osculating Conics" [Journal and Proceedings, Asiatic Society of Bengal (New Series), Vol. IV, No. 4, 1908] is

$$(3qs - 5r^2)\{(Y-y) - p(X-x)\}^2 + \{(Y-y)r - (X-x)(pr-3q^2)\}^2$$
$$= 18q^3\{(Y-y) - p(X-x)\},$$

or

$$\begin{aligned} &Y^2\{3qs - 4r^2\} + X^2\{p^2(3qs-5r^2) + (pr-3q^2)^2\} \\ &\quad -2XY\{p(3qs-5r^2) + r(pr-3q^2)\} \\ &\quad +2X\{p(y-px)(3qs-5r^2) + (pr-3q^2)[yr - x(pr-3q^2)] + 9pq^3\} \\ &\quad -2Y\{(y-px)(3qs-5r^2) + r[yr - x(pr-3q^2)] + 9q^3\} \\ &\quad +(y-px)^2(3qs-5r^2) + [yr - x(pr-3q^2)]^2 - 18q^3(px-y) = 0. \end{aligned}$$

Substituting for p, q, r and s their values from formulae (9) we have the co-efficient of Y^2

$$= 3qs - 4r^2 = \underset{x=\infty}{\mathrm{Lt}} \left[\frac{144}{x^{10}} (\gamma\epsilon - \delta^2) + \text{higher powers of } \frac{1}{x} \right];$$

the co-efficient of X^2

$$=p^2(3qs-5r^2)+(pr-3q^2)^2$$

$$=\underset{x=\infty}{\mathrm{Lt}}\left[\frac{144}{x^{10}}(\alpha\gamma^3+\alpha^2\gamma\epsilon-\alpha^2\delta^2)+\text{higher powers of }\frac{1}{x}\right];$$

the co-efficient of XY

$$=-2\{p(3qs-5r^2)+r(pr-3q^2)\}$$

$$=\underset{x=\infty}{\mathrm{Lt}}\left[\frac{144}{x^{10}}(-\gamma^3-2\alpha\gamma\epsilon+2\alpha\delta^2)+\text{higher powers of }\frac{1}{x}\right];$$

the co-efficient of X

$$=2\{p(y-px)(3qs-5r^2)+(pr-3q^2)[yr-x(pr-3q^2)]+9pq^3\}$$

$$=\underset{x=\infty}{\mathrm{Lt}}\left[\frac{144}{x^{10}}(2\gamma\epsilon\alpha\beta-2\delta^2\alpha\beta-\gamma^2\delta\alpha+\gamma^3\beta)+\text{higher powers of }\frac{1}{x}\right];$$

the co-efficient of Y

$$=-2\{(y-px)(3qs-5r^2)+r[yr-x(pr-3q^2)+9q^3\}$$

$$=\underset{x=\infty}{\mathrm{Lt}}\left[\frac{144}{x^{10}}(\gamma^2\delta-2\beta\gamma\epsilon+2\beta\delta^2)+\text{higher powers of }\frac{1}{x}\right];$$

and the absolute term

$$=(y-px)^2(3qs-5r^2)+[yr-x(pr-3q^2)]^2-18q^3(px-y)$$

$$=\underset{x=\infty}{\mathrm{Lt}}\left[\frac{144}{x^{10}}(\gamma^4-\gamma^2\delta\beta+\gamma\epsilon\beta^2-\delta^2\beta^2)+\text{higher powers of }\frac{1}{x}\right].$$

Thus the equation to the conic osculating the curve (1) at infinity reduces to

$$\begin{aligned}Y^2(\gamma\epsilon-\delta^2)&+X^2(\alpha\gamma^3+\alpha^2\gamma\epsilon-\alpha^2\delta^2)-XY(\gamma^3+2\alpha\gamma\epsilon-2\alpha\delta^2)\\&+X(2\alpha\beta\gamma\epsilon-2\alpha\beta\delta^2-\alpha\gamma^2\delta+\beta\gamma^3)-Y(2\beta\gamma\epsilon-\gamma^2\delta-2\beta\delta^2)\\&+\gamma^4-\gamma^2\delta\beta+\beta^2\gamma\epsilon-\beta^2\delta^2=0. \qquad \ldots \quad (10)\end{aligned}$$

This can also be written as

$$(Y-\alpha X-\beta)\{(\gamma\epsilon-\delta^2)Y-(\alpha\gamma\epsilon-\alpha\delta^2+\gamma^3)X+\gamma^2\delta-\beta\gamma\epsilon+\beta\delta^2\}+\gamma^4=0 \qquad \ldots \quad (11)$$

which shows that the osculating conic of a curve for a point at infinity is a hyperbola of which one asymptote is $Y-\alpha X-\beta=0$, and the other is

$$(\gamma\epsilon-\delta^2)Y-(\alpha\gamma\epsilon-\alpha\delta^2+\gamma^3)X+\gamma^2\delta-\beta\gamma\epsilon+\beta\delta^2=0. \qquad \ldots \quad (12)$$

The angle between the two asymptotes is given by

$$\tan\theta = \frac{\gamma^3}{(1+\alpha^2)(\gamma\epsilon-\delta^2)+\alpha\gamma^3}. \qquad \dots \quad (13)$$

If γ vanishes the two asymptotes coincide; the osculating conic degenerates into the asymptote counted twice and therefore the point at infinity becomes a point of inflexion Hence the condition that the point at infinity may be a point of inflexion is $\gamma=0$.

The co-ordinates of the centre of the osculating conic at any point x,y of a curve are

$$X = x - \frac{3qr}{3qs-5r^2}, \qquad Y = y - \frac{3q(pr-3q^2)}{3qs-5r^2}.$$

(*See General Theory etc., loc. cit.*); the limiting values of these when x and y tend to infinity, as obtained by using the formulae (9) are

$$X = \frac{\delta}{\gamma}, \qquad Y = \frac{\beta\gamma+\alpha\delta}{\gamma}. \qquad \dots \quad (14)$$

3. —We shall now find the equation of the equilateral hyperbola osculating the given curve at infinity. The general equation of the osculating equilateral hyperbola is

$$\{(X-x)^2-(Y-y)^2\}(2pr-3q^2)-2(X-x)(Y-y)\{(1-p^2)r+3pq^2\}$$
$$+6\{(Y-y)-(X-x)p\}q(1+p^2)=0,$$

(*see General Theory, etc, loc. cit.*) which can be written as

$$(X^2-Y^2)(2pr-3q^2)-2XY\{(1-p^2)r+3pq^2\}$$
$$-2X\{x(2pr-3q^2)-yr(1-p^2)-3ypq^2+3pq(1+p^2)\}$$
$$+2Y\{y(2pr-3q^2)+x(1-p^2)r+3xpq^2+3q(1+p^2)\}$$
$$+(x^2-y^2)(2pr-3q^2)-2xy\{(1-p^2)r+3pq^2\}+6(px-y)q(1+p^2)=0.$$

Now substituting the values of p, q, r and s from formulae (9) we have in the limit

$$2pr-3q^2 = \underset{x=\infty}{Lt}\left[-\frac{12\alpha\gamma}{x^4}+\text{higher powers of } \frac{1}{x}\right];$$

$$(1-p^2)r+3pq^2 = \underset{x=\infty}{Lt}\left[-\frac{6\gamma(1-\alpha^2)}{x^4}+\text{higher powers of } \frac{1}{x}\right];$$

$$x(2pr-3q^2)-yr(1-p^2)-3ypq^2+3pq(1+p^2)$$

$$=\underset{x=\infty}{Lt}\left[-\frac{6}{x^4}(\alpha\delta+\beta\gamma-\alpha^2\beta\gamma-\alpha^3\delta)+\text{higher powers of }\frac{1}{x}\right];$$

$$y(2pr-3q^2)+x(1-p^2)r+3xpq^2+3q(1+p^2)$$

$$=\underset{x=\infty}{Lt}\left[-\frac{6}{x^4}\{\delta(1+\alpha^2)+2\alpha\beta\gamma\}+\text{higher powers of }\frac{1}{x}\right];$$

and finally

$$(x^2-y^2)(2pr-3q^2)-2xy\{(1-p^2)r+3pq^2\}+6(px-y)q(1+p^2)$$

$$=\underset{x=\infty}{Lt}\left[\frac{12}{x^4}\{\alpha\beta^2\gamma+5\alpha^2\beta\delta-\alpha^2\gamma^2-4\alpha^3\delta-\gamma^2+4\alpha\delta-3\beta\delta\}\right.$$

$$\left.+\text{ higher powers of }\frac{1}{x}\right].$$

Thus the equation to the equilateral hyperpola osculating the given curve at infinity is

$$(Y^2-X^2)\alpha\gamma+XY\gamma(1-\alpha^2)+X(\alpha\delta-\beta\gamma+\alpha^2\beta\gamma+\alpha^3\delta)$$

$$-Y(\delta+\alpha^2\delta+2\alpha\beta\gamma)+\alpha\beta^2\gamma-\alpha^2\gamma^2+\alpha^2\beta\delta-\gamma^2+\beta\delta=0, \quad \ldots \quad (15)$$

which can also be written in the form

$$(Y-\alpha X-\beta)(\alpha\gamma Y+\gamma X-\delta-\alpha\beta\gamma-\alpha^2\delta)-\gamma^2(1+\alpha^2)=0. \quad \ldots \quad (16)$$

If a be the length of the semi-axis of the osculating equilateral hyper bola

$$a^2=\frac{27q^4(1+p^2)^{\frac{3}{2}}}{\{(pr-3q^2)^2+r^2\}^{\frac{3}{2}}}$$

(*General Theory, loc. cit.*), and this in the limit becomes

$$=2\gamma\left(1+\frac{1}{\alpha^2}\right)^{\frac{3}{2}}. \quad \ldots \quad (17)$$

Again the co-ordinates of the centre of the osculating equilateral hyperbola are (*See Locus Cit.*)

$$X=x+\frac{3qr(1+p^2)}{(pr-3q^2)^2+r^2},$$

$$Y=y+\frac{3q(pr-3q^2)(1+p^2)}{(pr-3q^2)^2+r^2};$$

the limiting values of these when x and y become infinite are

$$\left.\begin{aligned} X &= \frac{\delta}{\gamma}, \\ Y &= \frac{\beta\gamma+\alpha\delta}{\gamma}. \end{aligned}\right\} \quad \cdots \quad (18)$$

4. We have seen that five consecutive points of intersection of the curves (1) and (2) will be at infinity if the relations (3)—(*i*), (*ii*), (*iii*), (*iv*) and (*v*) hold. Another consecutive point will be at infinity if ϕ be determined so as to make the co-efficient of x^{n-5} vanish, that is, if

$$\phi f'_n + (\gamma\delta+\beta\epsilon)f''_n + \frac{1}{2}(\beta^2\delta+\gamma^2\beta)f'''_n + \frac{1}{6}\beta^3\gamma f^{iv}_n$$

$$+\frac{\beta^5}{120} f^{v}_n + \epsilon f'_{n-1} + \left(\frac{\gamma^2}{2}+\beta\delta\right)f''_{n-1} + \frac{1}{2}\beta^2\gamma f'''_{n-1} + \frac{\beta^4}{24} f^{iv}_{n-1}$$

$$+\delta f'_{n-2} + \beta\gamma f''_{n-2} + \frac{\beta^3}{6} f'''_{n-2} + \gamma f'_{n-3} + \frac{\beta^2}{2} f''_{n-3}$$

$$+\beta f'_{n-4} + f_{n-5} = 0\,;$$

α, β, γ, δ and ϵ being all known ϕ can be determined from this equation.

Since the curves (1) and (2) have now six points at infinity common the values of p, q, r, s and t for the two curves are the same at infinity. Hence

$$t = \mathop{\mathrm{Lt}}_{x=\infty} \left\{ -\frac{120\gamma}{x^6} - \frac{720\delta}{x^7} - \frac{2520\epsilon}{x^8} - \frac{6720\phi}{x^9} \right\}.$$

Now the condition for a sextactic point is

$$40r^3 - 45qrs + 9q^2t = 0,$$

(*General Theory*, *loc. cit.*) ; the limiting value of this when x tends to infinity is

$$\gamma^2\phi + 2\delta^3 - 3\gamma\delta\epsilon = 0. \quad \cdots \quad (18)$$

This is the condition that the point at infinity may be a sextactic point without being a point of inflexion.

5. Theorem : *All conics of four pointic contact at infinity to a given curve have a common centre.*

We have seen that the co-ordinates of the centre of the osculating conic at infinity as also of the osculating equilateral hyperbola are the same and involve only the constants α, β, γ and δ, Whence we can conclude that all conics of four pointic contact at a given point at infinity have the same centre. It is easy to deduce the same result from geometrical considerations as follows.

The director circles of all conics of four pointic contact at a given point of a curve form a co-axial system of which the limiting points are the given point of contact and the centre of the osculating equilateral hyperbola. (*See General Theory, loc. cit.*) Now if one of the limiting points be a point at infinity, the system of co-axial circles becomes concentric, having for the common centre the other limiting point. The system of conics of which the director circles are concentric are necessarily concentric.

We may call this common centre *a limiting centre* of the algebraic curve.

6 Theorem : *The three limiting centres of a cubic with three distinct real asymptotes are collinear.*

Let the cubic be

$$(y-a_1x-\beta_1)(y-a_2x)y+ly+mx+n=0$$

We have

$$f_3(a)=(a-a_1)(a-a_2)a=a^3-(a_1+a_2)a^2+a_1a_2a,$$

$$f'_3(a)=3a^2-2a(a_1+a_2)+a_1a_2,$$

$$f''_3(a)=6a-2(a_1+a_2),$$

$$f'''_3(a)=6\ ;$$

$$f_2(a)=-\beta_1(a-a_2)a,$$

$$f'_2(a)=-\beta_1(2a-a_2),$$

$$f''_2(a)=-2\beta_1\ ;$$

$$f_1(a)=la+m, \qquad f'_1(a)=l$$

and $f_0(a)=n.$

As the roots of $f_3(a)=0$ are a_1, a_2 and 0, the values of the several functions corresponding to these roots are given by

	a_1	a_2	0
f_3	0	0	0
f'_3	$a_1(a_1-a_2)$	$-a_2(a_1-a_2)$	a_1a_2
f''_3	$2(2a_1-a_2)$	$-2(a_1-2a_2)$	$-2(a_1+a_2)$
f'''_3	6	6	6
f_2	$-\beta_1(a_1-a_2)a_1$	0	0
f'_2	$-\beta_1(2a_1-a_2)$	$-\beta_1a_2$	β_1a_2
f''_2	$-2\beta_1$	$-2\beta_1$	$-2\beta_1$
f_1	la_1+m	la_2+m	m
f'_1	l	l	l
f_0	n	n	n

The values of β,γ,δ corresponding to the three values a_1, a_2 and 0 of a calculated with the help of the above table are

a	a_1	a_2	0
β	β_1	0	0
γ	$-\dfrac{la_1+m}{a_1(a_1-a_2)}$	$\dfrac{la_2+m}{a_2(a_1-a_2)}$	$-\dfrac{m}{a_1a_2}$
δ	$\dfrac{a_1\beta_1(la_1+m)-(a_1-a_2)(na_1-m\beta_1)}{a_1^2(a_1-a_2)^2}$	$\dfrac{n(a_1-a_2)-\beta_1(la_2+m)}{a_2(a_1-a_2)^2}$	$\dfrac{m\beta_1-na_1}{a_1^2a_2}$

Now the co-ordinates of the centre of an osculating conic at infinity are $\frac{\delta}{\gamma}$, $\frac{\gamma\beta+\alpha\delta}{\gamma}$. Hence the three centres will be collinear if the determinant

$$\begin{vmatrix} \delta_1 & \gamma_1\beta_1+\alpha_1\delta_1 & \gamma_1 \\ \delta_2 & \gamma_2\beta_2+\alpha_2\delta_2 & \gamma_2 \\ \delta_3 & \gamma_3\beta_3+\alpha_3\delta_3 & \gamma_3 \end{vmatrix}$$

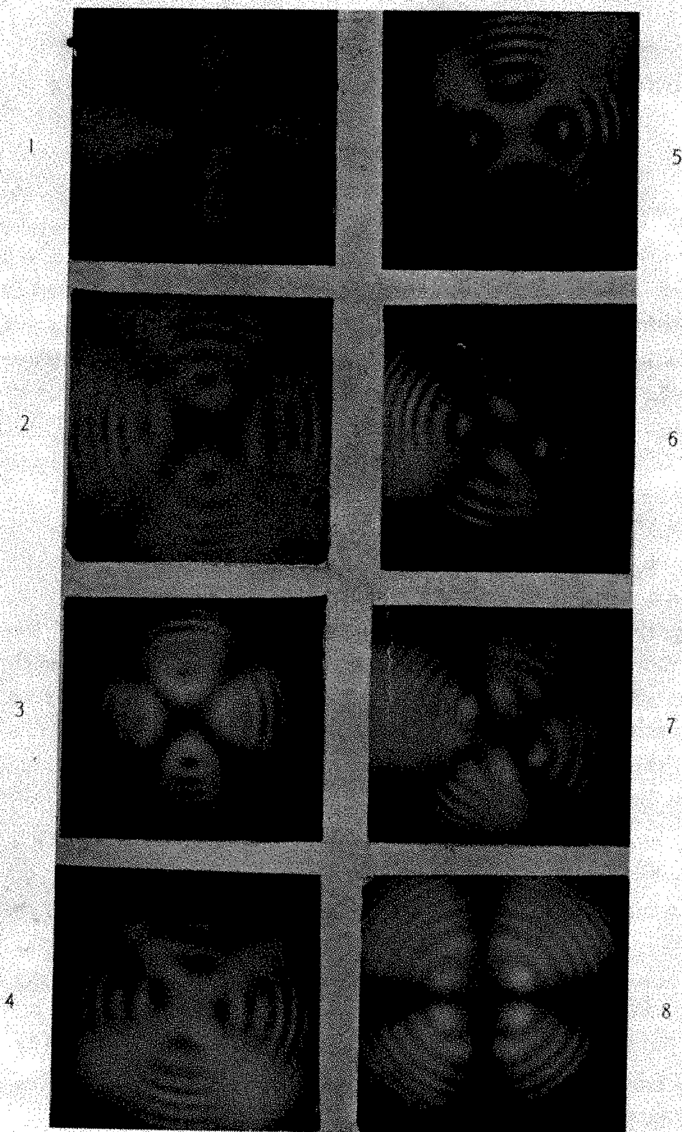
vanishes. On substituting the values of α_1, β_1, γ_1, δ_1, α_2, β_2,... from the above table the determinant becomes

$$= \frac{1}{(\alpha_1-\alpha_2)^3\alpha_1^3\alpha_2^2}\, \mathrm{D}$$

where D

$$\begin{aligned}
&=(m\beta_1-n\alpha_1)\{(l\alpha_2+m)[\alpha_2\beta_1(l\alpha_1+m)-(\alpha_1-\alpha_2)(n\alpha_1-m\beta_1)] \\
&\qquad +\alpha_2(l\alpha_1+m)[n(\alpha_1-\alpha_2)-\beta_1(l\alpha_2+m)]\} \\
&\quad -m\{\beta_1(l\alpha_1+m)\alpha_2[n(\alpha_1-\alpha_2)-\beta_1(l\alpha_2+m)]-[n(\alpha_1-\alpha_2)-\beta_1(l\alpha_2+m) \\
&\qquad \times[\alpha_2\beta_1(l\alpha_1+m)-(\alpha_1-\alpha_2)(n\alpha_1-m\beta_1)]]\} \\
&=(m\beta_1-n\alpha_1)\{-(l\alpha_2+m)(n\alpha_1-m\beta_1)+\alpha_2 n(l\alpha_1+m)\}(\alpha_1-\alpha_2) \\
&\qquad -m[n(\alpha_1-\alpha_2)-\beta_1(l\alpha_2+m)]\{\alpha_2\beta_1(l\alpha_1+m)-\alpha_2\beta_1(l\alpha_1+m) \\
&\qquad +(\alpha_1-\alpha_2)(n\alpha_1-m\beta_1)\} \\
&=(m\beta_1-n\alpha_1)(\alpha_1-\alpha_2)\{\alpha_2 n(l\alpha_1+m)-(n\alpha_1-m\beta_1(l\alpha_2+m)\} \\
&\qquad +(m\beta_1-n\alpha_1)(\alpha_1-\alpha_2)m\{n(\alpha_1-\alpha_2)-\beta_1(l_2\alpha+m)\} \\
&=(m\beta_1-n\alpha_1)(\alpha_1-\alpha_2)\{\alpha_2 n(l\alpha_1+m)-(n\alpha_1-m\beta_1)(l\alpha_2+m) \\
&\qquad +mn(\alpha_1-\alpha_2)-m\beta_1(l\alpha_2+m)\} \\
&=(m\beta_1-n\alpha_1)(\alpha_1-\alpha_2)\{\alpha_2 n(l\alpha_1+m)-n\alpha_1(l\alpha_2+m)+mn(\alpha_1-\alpha_2)\} \\
&=(m\beta_1-n\alpha_1)(\alpha_1-\alpha_2)\{ln\alpha_1\alpha_2+mn\alpha_2-ln\alpha_1\alpha_2-mn\alpha_1+mn\alpha_1-mn\alpha_2\} \\
&=0
\end{aligned}$$

which proves the theorem.



Illustrating the distorted " Rings and Brushes " as observed through a twin Crystal.

The Non-radiating Electronic Orbits and the Normal Zeeman Triplet.

By

Brajendranath Chuckerbutti.

The importance of Bohr's atom-constitution lies in the fact that it introduces a connexion between h, the "Wirkungsquantum" of Planck and radiation. The conception of the orbital rotation of the electron was made long before by Langevin and others to explain the idea of molecular magnets and the consequent magnetism of bodies. Thus, according to the electronic theory, in a dia-magnetic substance each molecule is supposed to contain many such orbits and their action at an external point is neutralised on account of the fact that the direction of revolution is different in different orbits; whereas in a para-magnetic substance, the neutralisation is not perfect there being a resultant at the external point. Bohr, however, was guided by the study of the known character of spectral lines of hydrogen and helium together with the conviction that the act of emitting electromagnetic radiation by an electronic constituent of an atom must be of explosive nature. To arrive at his theory, Bohr* has made some very fundamental assumptions the important ones being:—

(*a*) There are "stationary states" of the atomic system in which there is no emission of radiation.

(*b*) Transition of an electron between any two "stationary states" corresponds to gain or loss of energy and the consequent emission or absorption of energy radiation, the frequency of which is given by the relation

$$h\nu = A_1 - A_2 \qquad \dots \quad (1)$$

where ν is the frequency of the radiation and A_1, A_2 are the energies corresponding to the two "stationary states."

(*c*) Dynamical equilibrium of the system in "stationary states" is governed by the laws of electrostatics, *viz.*,

$$\frac{mv^2}{a} = \frac{F}{a^2} \qquad \dots \quad (2)$$

F being a function of the orbital radius a.

* N. Bohr, *Phil. Mag.* 1913, 1914 and 1915.

(*d*) The frequency "ν" of the homogeneous radiation emitted during transition is equal to half the frequency of revolution "n" of the electron in its final "stationary state," and the energy of the simplest system containing an electron revolving about a positive nucleus is determined by the relation

$$W=\tfrac{1}{2}\tau hn, \quad \tau=1,2,3\ldots\text{etc.}$$

The assumption of the stationary non-radiating orbits has obtained direct experimental support. Einstein and Haas* have succeeded in detecting a rotational mechanical effect produced when an iron bar is magnetised and have measured it. Their results agree very closely with those to be obtained on the assumption that the magnetisation of iron is due to rotating electrons and as pointed out by Einstein and Haas, these experiments indicate very strongly that electrons can rotate in atom without emission of energy radiation.

The velocity of electrons in "stationary states" is so great that it seems possible to explain the magnetic properties of elements on the basis of molecular currents. If i be the current generated by the electron revolving with frequency n in an orbital path of radius a, the magnetic moment is

$$M=\pi a^2 i=\pi a^2 en \qquad \ldots \quad (3)$$

From Bohr's assumptions *viz.*, $W=\tfrac{1}{2}ma^2n=\tfrac{1}{2}\tau hn$

$$M=h\cdot\frac{\tau\cdot e}{4\pi m}. \qquad \ldots \quad (4)$$

Thus, the magnetic moment of the electron in its stationary orbit is proportional to Planck constant. Hence, knowing the value for "a" we can calculate the values for the susceptibility "k." But the difficulty in varifying these values are enormous for the magnetic susceptibility of elements in their atomic state is not known, the experimental activities so long being confined to the consideration of molecular state only. Neither is it possible to calculate the susceptibility in the molecular state, as Bohr's theory of moleculés is not so definite as that of atoms.

The importance of Bohr's theory is further enhanced by the exactness with which it can explain the Balmer and other simple spectral series. The constant of the Balmer series in hydrogen is known with the great precision attained in all wavelength determinations and has the value $N=3\cdot290\times10^{15}$ From Bohr's equations this comes up to

$$N=\frac{2\pi^2e^2E^2m}{h^3}.$$

For hydrogen, $$N=\frac{2\pi^2e^4m}{h^3}.$$

* Einstein and Haas—Verh. D. Phys. Ges XVII, page 152, 1915.

Substituting the values of the constants, *viz.*,

$$e=4\cdot774\times10^{-10}\text{(Millikan, Proc. Natl. Acad. April, 1917)}$$

$$h=6\cdot56\times10^{-27}\text{(Millikan, Phys. Rev. Page 362, 1916)},$$

we get, $$N=3\cdot294\times10^{15}.$$

This agreement contributes the most extra-ordinary justificatian of the theory of non-radiating electronic orbits.

The radii of the stable orbits for hydrogen are given from Bohr's assumption as

$$a=\frac{\tau^2h^2}{4\pi^2me^4}\quad\text{where }\tau=1,2,3\text{ etc.}$$

i.e. the radii of non-radiating stable orbits are proportional to the square of the natural numbers. Taking normal hydrogen to be that in which the electron is on the innermost orbit, $2a$, the diameter of the normal hydrogen atom comes out to be $1\cdot1\times10^{-8}cm.$ The best determination for the diameter of the hydrogen molecule yields $2\cdot2\times10^{-8}cm$ in extra-ordinary close agreement with the prediction from Bohr's theory.

Let us consider in a diamagnetic gas an atom with an electronic orbit of radius "a," the electron revolving with a velocity v and suppose a magnetic field of strength H is applied perpendicular to the plane of the orbit. In considering the action of the magnetic field upon the stationary orbits, we must remember that on account of Bohr's assumption of constancy of angular momentum the velocity of revolution will remain the same after as before the application of the magnetic field. In other words, the action of the external field will simply be to alter the frequency of revolution and the radius of the orbit, which quantities will increase or decrease according to the direction of the field and the direction of rotation of the electron. Thus, in place of a single stationary orbit, we shall be in a position to conceive of three orbits, one on each side of the original orbit, the energy corresponding to which will be supplied by the applied external magnetic field. So that so long as the magnetic field is on, we have two more stationary orbits satisfying the condition that any electron jumping into them from any external orbit, will emit a radiation the frequency of which may be calculated from Bohr's assumptions. Hence, to arrive at an expression for the frequency difference between normal Zeeman triplets, let us proceed as follows :

From dynamical conditions

$$\frac{mv^2}{a}=\frac{F}{a^2}.$$

As the magnetic field is applied, we have

$$\frac{mv^2}{(a+da)} = \frac{F}{(a+da)^2} \pm Hev$$

where $(a+da)$ is the new radius

or $$\frac{mv^2}{(a+da)^2} - \frac{mv^2 a}{(a+da)^2} = \pm \frac{Hev}{a+da}.$$

Since $$v = 2\pi na = 2\pi n_1(a+da),$$

where n_1 is the frequency of the electron in the changed orbits,

we have $$n_1 - \frac{an_1}{a+da} = \pm \frac{He}{2\pi m}.$$

Then, since $$da = -\frac{v}{2\pi}\frac{dn}{n^2},$$

$$= -a\frac{dn}{n}$$

$$= a\frac{n_1 - n}{n},$$

we have $$n_1 - n = \pm \frac{He}{2\pi m}. \qquad \ldots (5)$$

Since the frequencies of the emitted radiation are half of those of the electron in the final stationary states, the difference of frequencies of radiation corresponding to the two hypothetical orbits will be

$$\nu_1 - \nu_2 = \pm \frac{He}{2\pi m} \qquad \ldots (6)$$

and hence for the frequency difference of the two outer-most component of the normal triplet, we have

$$\nu_1 - \nu_2 = \pm \frac{He}{2\pi m}. \qquad \ldots (7)$$

This expression is the same as that obtained according to Lorentz theory in which the restoring force varies with the distance.

In the foregoing pages, some consequences of Bohr's atom-constitution are discussed in the light of conclusions arrived at experimentally and finally an attempt is made to arrive at the expression for the frequency difference between the two outer-most components of the normal triplet. The arguments however, apply to the case of diamagnetic gases alone, where the action of the internal magnetic field is out of consideration the different components cancelling one another

On the Optical Analogue of the Whispering Gallery Effect.*

By

Bidhubhusan Ray.

1. *Introduction.*

In a paper published in the Proc. Roy. Inst. (1904) Lord Rayleigh described some interesting acoustical phenomena, which he observed at the Whispering gallery of St. Paul's cathedral in London. By using a high pitch source of sound and a high pressure sensitive flame receiver, Lord Rayleigh found that the sound vibrations have a tendency to cling to a concave surface so much so, that a narrow obstacle placed close to the surface is sufficient to stop mostly, if not entirely, the transmission of vibrations. In India, the Whispering gallery effect may be observed in the "Gol Gumbuz" of Bizapur.

Lord Rayleigh† gave a general explanation of the phenomena using the principle of geometrical optics and in a later paper‡ attempted an explanation of the same phenomena from the wave theory. His solution shows that in the neighbourhood of a concave reflecting wall, the intensity of vibrations becomes exceedingly pronounced. In 1914 Lord Rayleigh§ again reverted to the same problem and obtained the result that the wave as a whole creeps along the surface and that the wave front is perpendicular to the concave wall.

Scott Russell has pointed out that in the case of a water wave proceeding almost parallel to a curved surface, reflection fails to occur, the line of wave-crest near the wall, setting itself normal to the wall, so that the propagation is along the wall itself.

* This paper was read at the Science convention held at the Indian Association for the cultivation of Science in March, 1919.

† Theory of Sound, Vol. 2.

‡ Phil. Mag., Vol. 20, 1910.

§ Phil. Mag., 1914 or Complete Works, Vol. VI.

The most important practical demonstration of the gallery effect was made by Burton and Kilby,* who showed by means of very short sound waves produced by spark discharge and recorded by dust particles, that the normal striae about an inch long occur all along the interior of the curved boundary, showing that the air motions are along the boundary itself. At the far end of the figure, it is noticeable that the effect partly branches off to the other obstacle." These experiments seem to confirm Lord Rayleigh's observations.

It occurred to the present writer that it would be of interest to study the corresponding optical phenomena and some very striking results were obtained. Although as in the Whispering gallery phenomena most of the effects were confined within a narrow region from the reflecting surface, on closer observation it was observed that in the optical case, we obtain instead of a single belt of maximum intensity a succession of belts of alternately great and small intensities.

2. *Experimental method and observations.*

The experimental arrangements adopted to observe the effect due to a light source placed on a curved reflecting surface are as follows. A strip of a plane mirror about 3 ft. in length and 3 inches in breadth was supported on wooden edges placed near its ends. Two light wooden bridges, placed on the mirror near its two ends and equally loaded as desired, enabled the mirror to be bent into a curved surface of variable radius of curvature. By noting the depression of a pin head attached to the middle point of the mirror radially, the radius of curvature of the mirror was easily found out. A fine razor edge placed on the mirror near one end formed an exceedingly fine slit between it and the reflecting surface. On this slit, light from a Cooper-Hewitt mercury lamp, filtered through Quinine Sulphate solution was focussed. A good photographic lens is placed at a distance of its focal length from the edge and the effects are observed by means of an eye piece from a long distance. By exposing photographic plates properly placed in place of the eye-piece the effects were photographed.

When the mirror is not bent, only a general illumination of the field is observed from the far end of the mirror when a small weight is placed on the mirror and a slight curvature is imposed on it, a bright band of increased luminosity flashing from the surface of the mirror is observed. When the curvature is further increased, this band contracts in width and a second band appears in the field of view separated from

* Phil. Mag., Vol. 24, 1912.

the first by a dark band. Still more increase of curvature brings a third band preceded by another dark band and so on. As the curvature of the mirror is increased, the number of bands and their sharpness also increase and their width decreases.

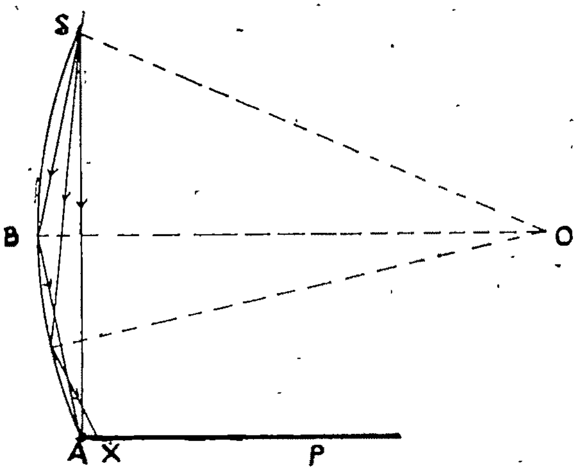
These features are illustrated in the few photographs reproduced in the Plate.

Fig. 1 (plate) illustrates the luminosity observed with no weight not even the bridge placed upon the mirror. Even the image of the slit formed by the knife-edge is not to be seen without any trace of the features to be observed in the subsequent photographs which are taken with the weights on. The luminous band observed when only the wooden bridge is placed upon the mirror is shown in Fig. 2. The light is heaped as it were near the concave surface increasing the brilliancy of a portion just in contact with the mirror, far above the rest of the field. Fig. 3 (plate) illustrates the effect when 2 kg. weight is put on the bridge. The bands are clearly seen with all the peculiarities mentioned before. The last band is very indistinct and the edges are sharper than the one recorded in Fig. 2 Figures (4, 5 and 6) record the luminous effects, observed with 4, 5 and 9 kg. weights respectively placed upon the bridge. The number of bands is considerably increased. In the last Fig. as many as 22 bands are easily recognised. It should be pointed out here that the different bands are not of equal intensity, the difference of intensity is very prominent near the edge of the mirror. This feature is also apparent from the photographs (see Fig. 6) attached to this paper.

3. *Theoretical consideration.*

The explanation of these effects can be obtained in a general way by considering the simple reflection laws of geometrical optics. In order to find the intensity of light at any point near the edge of the mirror we must consider the rays, which reach there not only after one but also after two, three and more reflections. It is also evident that at any point near the surface, not only one but two rays after one reflection will pass through. Similarly we have two rays after two reflections passing through the same point and so on. It is to the interference of these large number of rays that we should seek for an explanation of the phenomena observed.

Indeed we have here a large number of caustics formed by rays suffering one, two, three or more reflections. Suppose, the caustic formed by one reflection cuts AP at a point X. Then it is obvious that no other ray after one reflection can reach AP beyond X.



At any point between AX two rays issuing from S and having different angles of incidence, after one reflection meet and interfere with each other. If the curvature of the mirror is large, the caustic formed by two reflections is formed, which cuts AP in some point between A, X and so on for the third and higher order caustics formed by larger number of reflections. If the curvature of the mirror is small, the distance within which the effects of the different caustics, formed by two or more reflections, are to be considered, is small and can be neglected. We now consider the simplest case when only a single reflection occurs.

The distance AX($=x$)

$$=r\{\sin\alpha-\sin(\theta-\alpha)\}\tan\left(\frac{3\theta}{2}-\alpha\right)-r\{\cos(\theta-\alpha)-\cos\alpha\}$$

(where r is the radius of curvature of the mirror, α is the semi angular aperture of the mirror and θ is the variable angle SOC.)

This when simplified becomes

$$x=r\{3r\theta-\theta^2-2a^2\}$$

we get therefore two values for θ, for any value of x. The path difference of the two rays which meet at any point depends on the angle of incidence and is greatest at the point A and when calculated, is easily found to be $4r\sin\frac{a}{2}-2r\sin a$

As we proceed from A to the point where the caustic cuts AP, this difference diminishes till at the point X, it assumes a zero value. If the

path difference is $m\lambda$ at A, we may expect about m bands near the edge of the mirror. The position of the next band will therefore be determined by rays having a path difference $(m-1)\lambda$ and meeting at a point Y (say). When the curvature is increased this path difference of the rays SBA and SA is also increased. Let $m'\lambda$ denote this path difference for a different configuration of the mirror then $m'>m$. The position of the next band for this configuration is at Y′ (say) and determined by a path difference $(m'-1)\lambda$. In this case evidently Y′ lies between A and Y This shows that the number of bands should increase and their width should decrease as the curvature is increased.

Similar arguments will hold for higher order caustics but the calculations will obviously become more and more complicated.

The difference of intensity of bands already pointed out may perhaps be explained as due to the superposition of caustics of second and higher orders.

The intensity is calculated by mean[illegible]general formula

$$I=\sum a_1^2+2\sum a_1 a_2 \cos\frac{2\pi}{\lambda}(\text{[illegible]}$$

for different values of x.

The path difference of the rays me[illegible] point A (after one reflection) is $4\,r\sin\frac{a}{2}-2r\sin a$, that is[illegible]ately and we should therefore expect to observe about 4[illegible]hich is actually the case. The fourth band is however very [illegible]ct. For one reflection, the intensity as calculated from this formula and also observed values of the intensity of the field (Photographed in Fig. 3) are given in the following table for comparison

m	Position of the dark band from the edge of the mirror.	
	observed	calculated
1	·0058 c.m.	·0061 c.m.
2	·0119 c.m	·0123 c.m.
3	·0174 c.m.	·0180 c.m.

The agreement between calculated and observed values is fairly satisfactory.

4. *Summary and Conclusion.*

The general explanation of the whispering gallery effect, that the wave as a whole creeps along the surface and the wave front is perpendicular to concave wall, has been given by the late Lord Rayleigh on the principles of the wave theory. These effects have been verified by Russell, Burton and Kilby.

The analogous optical case has been studied experimentally in the present paper. Instead of a single belt of maximum intensity close to a moderately curved reflecting surface, a succession of maximum belts has been observed. These maxima presenting themselves in the form of a series of interference bands when observed at the far end of a strip of bent mirror havi[illegible] light source on its surface at one end, and observed to undergo [illegible] fluctuations as the curvature of the mirror is increased. W[illegible]crease of curvature the number of bands and their sharpn[illegible] but their width decreases.* These peculiarities have bee[illegible] in this paper from the theory of geometrical optics.

The investigation w[illegible]t in the Palit Laboratory of Physics, and the writer wishe[illegible] Prof. C. V. Raman for his helpful interest during the pr[illegible] work.

* Similar observations [illegible] recently made by Raman and Sutherland at the St. Paul's cathedral for sound waves and their observations have been published in the Proc. Roy. Soc. Ser. A, Vol. 100, January, 1922.

On the dissipation of the energy of impact in the form of elastic waves

By

Nripendrakumar Majumdar, M.A.

From Hertz's theory of impact we know that the duration of impact is very small in absolute value; but it is very large compared with the time taken by elastic waves of deformation to traverse far into the impinging bodies. Hence we see that though the force acts over the compressed area and disturbance originates in it, neighbouring portions will be thrown into state of strain before the termination of the impact.

In this paper, I have investigated the elastic displacement, and energy propagated, in a body of considerable dimensions, with plane face, when a hard spherical ball drops upon it.

All the linear dimensions of the body being large compared with those of the area subjected to the pressure, we may as a first approximation regard the body as bounded by an infinite plane.

The force that acts on the body is the mutual pressure between the impinging bodies. I have supposed, as in Hertz's theory, that the pressure is equivalent to a single force, equal to the whole pressure between the bodies at any instant during the impact and is acting at the centre of compression. I have discussed the displacements before the termination of the impact.

Reference may be made to a paper* by Lord Rayleigh who has investigated the circumstances of the first appearaace of sensible vibrations in the case of two impinging spheres.

The centre of compression is taken to be the origin, the plane face being the plane $x=0$ and the axis of x penetrating into the body. Thus pressure P acts along the x-axis. The pressure P is zero, just before the impact, then it gradually increases, attaining its maximum value at the time of greatest compression; then it gradually diminishes and becomes

* Lord Rayleigh "On the production of vibrations by forces of Relatively Long Duration with Application to the Theory of Collisions," *Phil. Mag.*, Vol. XI, pp. 283-291 (1906). [Scientific Papers, Vol. V, pp. 292-299.]

again zero at the close of the impact. We know $P=k_2 a^{\frac{3}{2}}$ (see Love's Elasticity, pp. 196, 2nd Edition).

If we assume $P=k_2 a_1^{\frac{3}{2}} \sin \frac{\pi t}{2\tau}$, where a_1 is the value of a, at the time of greatest compression and 2τ is the duration of impact, time being measured from the beginning of the impact, we see that all the above three conditions are satisfied.

Thus $\quad P=A \sin pt$, where $A=k_2 a_1^{\frac{3}{2}}$, $p=\frac{\pi}{2\tau}$. $\quad \ldots \quad (1)$

The equations of motion are of the type

$$(\lambda+\mu)\left(\frac{\partial \Delta}{\partial x}, \frac{\partial \Delta}{\partial y}, \frac{\partial \Delta}{\partial z}\right)+\mu(\nabla^2 u, \nabla^2 v, \nabla^2 w)+\rho(X,Y,Z)$$

$$=\rho\left(\frac{\partial^2 u}{\partial t^2}, \frac{\partial^2 v}{\partial t^2}, \frac{\partial^2 w}{\partial t^2}\right) \quad \ldots \quad (2)$$

In the present case $Y=Z=0$, $X=A \sin pt$.

Assuming $(u,v,w)=$gardient $\phi+$curl (F,G,H) and substituting in equations (2), we find a particular solution of them in the form : (see Lord Rayleigh, Theory of Sound, Vol. 2, §276)

$$u_1=\frac{A}{4\pi\rho}\frac{\partial^2 r^{-1}}{\partial x^2}\int_{\frac{r}{b}}^{\frac{r}{a}} t' \sin p\,(t-t')dt' + \frac{A}{4\pi\rho r}\left(\frac{\partial r}{\partial x}\right)^2$$

$$\left[\frac{1}{a^2}\sin p\left(t-\frac{r}{a}\right)-\frac{1}{b^2}\sin p\left(t-\frac{r}{b}\right)\right]+\frac{A}{4\pi\rho b^2 r}\sin p\left(t-\frac{r}{b}\right) \quad (3)$$

and two other similar expressions.

Here a and b are velocities of propagation of irrotational and equivoluminal waves respectively. Particular integrals come out, when simplified, in the form

$$\left.\begin{aligned} u_1&=\frac{A}{4\pi\rho}\left[\frac{1}{r^3}\left(\frac{3x^2}{r^2}-1\right)\frac{\phi_1(tr)}{p^2}+\frac{x^2}{r^3}\phi_2(tr)+\frac{1}{b^2 r}\sin p\left(t-\frac{r}{b}\right)\right]\\ v_1&=\frac{A}{4\pi\rho}\left[\frac{3xy}{r^5}\frac{\phi_1(tr)}{p^2}+\frac{xy}{r^3}\phi_2(tr)\right]\\ w_1&=\frac{A}{4\pi\rho}\left[\frac{3xz}{r^5}\frac{\phi_1(tr)}{p^2}+\frac{xz}{r_3}\phi_2(tr)\right]\end{aligned}\right\} \quad (4)$$

in which ,

$$\phi_1(tr)=\sin p\left(t-\frac{r}{b}\right)-\sin p\left(t-\frac{r}{a}\right)+\frac{pr}{b}\cos p\left(t-\frac{r}{b}\right)$$

$$-\frac{pr}{a}\cos p\left(t-\frac{r}{a}\right),$$

$$\phi_2(t\gamma)=\frac{1}{a^2}\sin p\left(t-\frac{r}{a}\right)-\frac{1}{b^2}\sin p\left(t-\frac{r}{b}\right)$$

and
$$\left(\frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ \frac{\partial}{\partial z}\right)\phi_1=-p^2(x,y,z)\phi_2. \qquad \dots \quad (5)$$

The complete solution of the equations of motion will be of the form

$$\left.\begin{aligned} u&=u_1+u_2\\ v&=v_1+v_2\\ w&=w_1+w_2\end{aligned}\right\}$$

where u_2, v_2, w_2 will be determined from the fact that the plane $x=0$ should be free from tractions. Thus

$$\left[\lambda\Delta+2\mu\frac{\partial u}{\partial x}\right]_{x=0}=0,\left[\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right]_{x=0}=0,\left[\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right]_{x=0}=0.$$

From these conditions we get, making use of the relations

$$\left(\frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ \frac{\partial}{\partial z}\right)\phi_1=-p^2(x,y,z)\phi_2,$$

$$\left.\begin{aligned} u_2&=-\frac{A}{4\pi\rho b^2}\ \frac{1}{r}\sin p\left(t-\frac{r}{b}\right)\\ v_2&=-\frac{A}{4\pi\rho}\left[\frac{6xy}{r^5}\ \frac{\phi_1}{p^2}+\frac{2xy}{r^3}\ \phi_2\right]\\ w_2&=-\frac{A}{4\pi\rho}\left[\frac{6xz}{r^5}\ \frac{\phi_1}{p^2}+\frac{2xz}{r^3}\ \phi_2\right]\end{aligned}\right\} \qquad \dots \quad (6)$$

Hence the complete solutions of the eqution of motion are, from (4) and (6)

$$\left.\begin{aligned} u&=\frac{A}{4\pi\rho}\left[\frac{1}{\gamma^3}\left(\frac{3x^3}{r^2}-1\right)\frac{\phi_1(tr)}{p^2}+\frac{x^3}{r^3}\phi_2(tr)\right] \\ v&=\frac{A}{4\pi\rho}\left[\frac{3xy}{r^5}\frac{\phi_1(tr)}{p^2}+\frac{xy}{r^3}\phi_2(tr)\right] \\ w&=\frac{-A}{4\pi\rho}\left[\frac{3xz}{r^3}\frac{\phi_1(tr)}{p^2}+\frac{xz}{r^3}\phi_2(tr)\right]\end{aligned}\right\} \quad \ldots \quad (7)$$

They evidently satisfy the remaining condition

$$\left[\lambda\Delta+2\mu\frac{\partial u}{\partial x}\right]_{x=0}=0.$$

Moreover, we find that the tractions calculated from u,v,w, over a spherical surface, with the origin as centre, are equivalent to a single force parallel to the axis of x.

Now the energy propagated at any instant is given by

$$2T=\rho\int_{r=0}^{at}\int_0^{\frac{\pi}{2}}\int_0^{2\pi}\left[\left(\frac{\partial u}{\partial t}\right)^2+\left(\frac{\partial v}{\partial t}\right)^2+\left(\frac{\partial w}{\partial t}\right)^2\right]r^2\sin\theta\, dr\, d\theta\, d\phi$$

integration being carried through a hemispherical volume, given by $r=at$, $u=0$, that is, though a volume up to which the disturbance has spread.

Neglecting higher inverse powers of r we get

$$2T=\frac{A^2}{16\pi^2\rho}\int_{r=0}^{at}\int_{\theta=0}^{\frac{\pi}{2}}\int_{\phi=0}^{\pi}\left[\frac{1}{a^4}\cos^2 p\left(t-\frac{r}{a}\right)+\frac{1}{b^4}\cos^2 p\left(t-\frac{r}{b}\right)\right.$$

$$\left.-\frac{2}{a^2b^2}\cos p\left(t-\frac{r}{a}\right)\cos p\left(t-\frac{r}{b}\right)\right]\sin^5\theta\cos^2\phi\, dr\, d\theta\, d\phi$$

$$=\frac{A^2p}{96\pi\rho a^3b^3}\left[\frac{2p}{b}(a^4+b^4)t+\frac{(a-b)^2(a^2+b^2+3ab)}{a+b}\sin 2pt\right.$$

$$\left.-\frac{8a^3b^2}{a^2-b^2}\sin p\frac{a-b}{b}t+a^3\sin 2p\frac{a-b}{b}t\right] \quad \ldots \quad (8)$$

But at the termination of the impact we have

$$t = 2\top = \frac{\pi}{p} \quad \text{from (1)}$$

$$\therefore \ pt = \pi.$$

Hence the energy propagated by the impact is given by (8) when π is written for pt,

$$\text{Thus} \quad 2\top = \frac{A^2 p}{96\pi\rho a^3 b_3} \left[\frac{2\pi}{b}(a^4+b^4) + \frac{8b^2a^3}{a^2-b^2} \sin\frac{\pi 9}{b} + a^3 \sin\frac{2\pi a}{b} \right] \quad (9)$$

Let us substitute the values of a, b, A and p in terms of elastic constants $(\lambda_1\mu_1, \lambda_2\mu_2)$ of the impinging bodies. We further suppose the relation $\lambda = \mu$ to hold.

$$\text{Now} \qquad a^2 = \frac{\lambda+2\mu}{\rho} = \frac{3\mu}{\rho} \qquad \text{and} \qquad b^2 = \frac{\mu}{\rho},$$

$$A = k_2 a_1^{\frac{3}{2}} \qquad \text{and} \qquad a_1 = \left(\frac{5}{x_1 k_2}\right)^{\frac{2}{5}} \left(\frac{v}{2}\right)^{\frac{4}{5}},$$

where v is the velocity of approach just before impact. See Love's Elasticity, 2nd Edition, pp. 192 and 197.

If m be the mass of the spherical ball

$$K_1 = \frac{1}{m},$$

since the mass of the ball is small compared with the block.

Making these substitutions we have from (9)

$$2\top = \frac{25\Gamma(\frac{9}{10})}{216\Gamma(\frac{2}{5})} \left(\frac{\mu_1\mu_2}{\mu_1+\mu_2}\right)^{\frac{6}{5}} \left(\frac{1}{150}\right)^{\frac{1}{5}} \left(\frac{1}{\pi}\right)^{\frac{7}{10}} \left(\frac{\rho}{\mu_2}\right)^{\frac{3}{5}} \frac{1}{\rho}$$

$$[20\pi + 12\sqrt{3}\sin\sqrt{3}\pi + 2\sqrt{3}\sin 2\sqrt{3}\pi] + mv^{\frac{13}{5}}$$

$$\doteq \frac{25}{216} \ \frac{\Gamma(\frac{9}{10})}{\Gamma(\frac{2}{5})} \left(\frac{\mu_1}{\mu_1+\mu_2}\right)^{\frac{8}{5}} \left(\frac{1}{\pi}\right)^{\frac{7}{10}} \left(\frac{\rho}{\mu_2}\right)^{\frac{1}{5}}$$

$$[20\pi + 12\sqrt{3}\sin\sqrt{3}\pi + 3\sqrt{3}\sin 2\sqrt{3}\pi] + mv^{\frac{13}{5}}$$

where μ_1 and σ are the rigidity and density of the ball and μ_2 and ρ are those for the block.

The above solution is invalidated when the solid is terminated. The problem has then to be tackled by a method of successive approximations. For example, if the solid is a plate bounded by $x=0$ and $x=h$ (say), then we seek to satisfy the condition of free traction on the boundary $x=h$ by the addition of three terms (u_3, v_3, w_3) being solutions of the elastic wave equations to the already existing displacements $(u_1+u_2, v_1+v_2, w_1+w_2)$. This will vitiate the condition at the surface $x=0$ to satisfy which we introduce three more terms namely (u_4, v_4, w_4) and so on. Each step in the proces will make the solution more and more complicated but the proceedure will ensure a rapid convergency of the solution.

STABILITY OF DIRIGIBLE BALOON

BY

NALINIKANTA BASU.

In the following discussion of the stability of a dirigible baloon the author has adopted certain suggestions and assumptions made by Professor Bryan in his determination of the Aeroplane stability. The cheif difficulty in the present problem lies in the assumption that it is legitimate to add the air resistance terms $-X_0-uX_u-vX_v-rX_r$, etc., to those dependant on the inertia of the displaced air. The validity of this can be tested by experiment and observations carried on actual airship motion.

When a solid moves through a fluid, its equations of motion may be written down by Hydrodynamical Principles. If T be the kinetic energy of the moving solid and u, v, w, p, q, r be its velocities; X,Y,Z,L,M,N be the component and moment of external forces, along and about three axes fixed in the solid and moving with it, (the external forces include the liquid pressure on the solid and gravity) the equations of motion of the solid may be written in the form

$$\frac{d}{dt}\cdot\frac{\partial T}{\partial u}-r\frac{\partial T}{\partial v}+q\frac{\partial T}{\partial w}=H+X$$

$$\frac{d}{dt}\cdot\frac{\partial T}{\partial v}-p\frac{\partial T}{\partial w}+r\frac{\partial T}{\partial u}=X$$

$$\frac{d}{dt}\cdot\frac{\partial T}{\partial w}-q\frac{\partial T}{\partial u}+p\frac{\partial T}{\partial v}=Z$$

$$\frac{d}{dt}\cdot\frac{\partial T}{\partial p}-r\frac{\partial T}{\partial q}+q\frac{\partial T}{\partial r}-w\frac{\partial T}{\partial v}+v\frac{\partial T}{\partial w}=L$$

$$\frac{d}{dt}\cdot\frac{\partial T}{\partial q}-p\frac{\partial T}{\partial r}+r\frac{\partial T}{\partial p}-u\frac{\partial T}{\partial w}+w\frac{\partial T}{\partial u}=M$$

$$\frac{d}{dt}\cdot\frac{\partial T}{\partial r}-q\frac{\partial T}{\partial t}+p\frac{\partial T}{\partial q}-v\frac{\partial T}{\partial u}-u\frac{\partial T}{\partial v}=N.$$

In finding T account will have to be taken of the inertia of the displaced air. In the case of a sphere moving in a perfect liquid we know that this inertia is equivalent to adding half the mass of the displaced liquid to the mass of the sphere. For a balloon in equilibrium the total weights of the airship and the displaced air are equal and hence the inertia of the air become of considerable importance. The inertia of the contents of the gas bag must also be taken into account. If we suppose the balloon moving in air to be represented by an ellipsoid moving in an incompressible medium we can find an expression for the kinetic energy by the hydrodynamical method. This expression will be owing to the (assumed) symmetry of the machine about a vertical plane, the sum of two homogenous quadratic functions, one of $U+u$, v, r and the other of w, p, q, when U is the velocity of the airship in the direction of the x-axis before the oscillations started. Hence (after Bryan)

$$
\begin{aligned}
2T &= Au^2+Bv^2+Rr^2+2Duv+2Evr+2Fru \\
&\quad +Cw^2+Pp^2+Qq^2+2D'wp+2E'pq+2F'qw \\
&= A(U+u)^2+Bv^2+Rr^2+2D(U+u)v+2Evr+2Fr(U+u) \\
&\quad +Cw^2+Pp^2+Qq^2+2D'wp+2E'pq+2F'qw.
\end{aligned}
$$

In considering the effects of gravity, we find that the weight of the airship is balanced by the buoyancy of the air according to the principles of Archimedes. In a state of equilibrium (at rest) the centres of gravity and buoyancy are in the same vertical line. If c be the distance between them, then when the axis of x is depressed from a horizontal position through an angle θ, and the machine subsequently turned through an angle ϕ about this axis couples an produced about the axes of

	x	y	z
of amount	$-cW\cos\theta\sin\phi$	0	$-cW\sin\theta$

and there are no component forces due to gravity along the axes. Consequently the forms depending on the action of gravity occur in the equations of rotation about the axes of x and z instead of in the equation of translation.

The principal obstacle in the way of a satisfactory treatment of the stability of a dirigible is the difficulty of making suitable assumptions regarding air resistance due to causes other than the inertia already mentioned. We shall assume that these effects are as in the case of aeroplane, represented to the first order by terms of the form

$$-X_0-uX_u-vX_v-rX_r$$

where X_0 is proportional to u^2 and the rest are proportional to U. (Bryan).

With these assumptions, the equations of motion become

$$A\frac{du}{dt}-Bvr+Cqw=H-X_0-vX_v-rX_r$$

$$B\frac{dv}{dt}-Cpw+A(U+u)r=-Y_0-uY_u-vY_v-rY_r$$

$$C\frac{dw}{dt}-A(U+u)q+Bpv=-Z_0-wZ_w-pZ_p-qZ_q$$

$$P\frac{dp}{dt}-Qqr+Rqr-(B-C)vw=-L_0-wL_w-qL_q$$
$$-pL_p-cW\cos\theta\sin\phi$$

$$Q\frac{dq}{dt}-(P-R)pr-(A-C)(U+u)w=-M_0-wM_w$$
$$-pM_p-qM_q$$

$$R\frac{dr}{dt}-(P-Q)pq-(A-B)(U+u)v=-N_0-uN_u$$
$$-vN_v-rN_r-CW\sin\theta.$$

If in these equations we neglect terms of the second order of small quantities when u, v, w, p, q, r are of the first order and U a difinite quantity we get

$$A\frac{du}{dt}+uX_u+vX_v+rX_r=H-X_0$$

$$uY_u+B\frac{dv}{dt}+vY_v+(AU+Y_r)r=-Y_0$$

$$C\frac{dw}{dt}+wZ_w+pZ_w+(AU+Z_q)q=-Z_0$$

$$P\frac{dp}{dt}+pL_p+qL_q+wL_w=-L_0-cW\cos\theta\sin\phi$$

$$Q\frac{dq}{dt}+qM_q+pM_p+(C-A+M_w)w=-M_0$$

$$R\frac{dr}{dt}+rN_r+uN_u+(N_v-A+B)v=-N_0-cW\sin\theta.$$

We rearrange the equations in two groups, the first group containing those (the 1st, 2nd and 6th) involving u, v, r, the second (consisting of the 3rd, 4th, 5th) involving w, p, q. We thus obtain the groups

$$A\frac{du}{dt}+uX_u+vX_v+rX_r=H-X_0$$

$$uY_u+B\frac{dv}{dt}+vY_v+(AU+Y_r)r=-Y_0$$

$$uN_u+\{(B-A)U+N_v\}v+R\frac{dr}{dt}+rN_r=-N_0-CW\sin\theta.$$

Initially $u=U$, $X_u=0$, $v=0$, $r=0$ then $H-X_0=0$

$$Y_u=0, \qquad Y_0=0$$

$$\theta=\theta_0 \quad N_u=0, \quad N_r=0-N_0-CW\sin\theta_0=0.$$

Then if $\theta=\theta_0+\epsilon$, $\sin\theta=\sin\theta_0+\epsilon\cos\theta_0$ where ϵ is a small angle.

Hence the 1st group becomes

$$\left(A\frac{d}{dt}+X_u\right)u+vX_v+rX_r=0$$

$$Y_u\cdot u+\left(B\frac{d}{dt}+Y_v\right)v+(AU+Y_r)r=0$$

$$N_u\cdot u+\{(B-A)U+N_v\}v+\left(R\frac{d}{dt}+N_r\right)r=-cW\epsilon\cos\theta_0.$$

To investigate this group of oscillations we assume u, v, r and ϵ to be proportional to $e^{\lambda t}$ so that

$$\frac{du}{dt}=\lambda u,\ \frac{dv}{dt}=\lambda v,\ \frac{dr}{dt}=\lambda r,\ \frac{d\epsilon}{dt}=\lambda\epsilon=r$$

the last equation gives $\lambda\epsilon=r$ or $\epsilon=\dfrac{r}{\lambda}$.

On transposing in the equations they become

$$(A\lambda+X_u)u+X_v\cdot v+X_r\cdot r=0$$

$$Y_u\cdot u+(B\lambda+Y_v)v+(AU+Y_r)r=0$$

$$N_u\cdot u-\{(A-B)U-N_v\}v+\left\{R\lambda+c\frac{W}{\lambda}\cos\theta_0+N_r\right\}r=0.$$

Eliminating u, v, r we get the determinant

$$\begin{vmatrix} A\lambda+X_u, & X_v, & X_r \\ Y_u, & B\lambda+Y_v, & AU+Y_r \\ N_u, & -\{(A-B)U-N_r\}v, & R\lambda+c\dfrac{W}{\lambda}\cos\theta_0+N_r \end{vmatrix}=0$$

Developing the determinant in powers of λ we get an equation of the fourth degree which we write

$$A_0\lambda^4+B_0\lambda^3+C_0\lambda^2+D_0\lambda+E_0=0$$

where $A_0=ABR,$

$$B_0=ABN_r+ARY_r+BRX_u,$$

$$C_0=ABcW\cos\theta_0+AY_vN_r+(X_uY_v-X_vY_u)R$$
$$+B(X_uN_r-X_rN_u)-A(AU+Y_r)\{N_v-(A-B)U\}$$

$$D_0=cW\cos\theta_0(BX_u+AY_v)+(X_uY_v-X_vY_u)N_r$$
$$-(AU+Y_r)(X_uN_v+X_vN_u)+X_r(Y_uN_v-Y_rN_u)$$
$$+(A-B)UN_v(X_uY_r-X_rY_v)+(A-B)AX_uU^2,$$

$$E_0=cW\cos\theta_0(X_uY_v-X_vY_u).$$

The conditions of stability require that all the four roots of the biquadratic equation in λ shall have their real part negative. This follows from the assumption that the small disturbance u, v, r etc., are all proportional to $e^{\lambda t}$ in a typical oscillation. If $\lambda_1\lambda_2\lambda_3\lambda_4$ are the roots of the biquadratic, the expressions for u, v, r take the form

$$a_1e^{\lambda_1 t}+a_2e^{\lambda_2 t}+a_3e^{\lambda_3 t}+a_4e^{\lambda_4 t}$$

$a_1a_2a_3a_4$ being constants determined by the initial conditions.

If any of the roots λ_1 is real and positive, a disturbance of the form $u=a_1e^{\lambda_1 t}$ will increase indifinitely with the time and steady motion will be unstable.

If on the other hand λ_1 is a real and negative quantity equal to $-k$, the expression $u=a_1e^{-kt}$ represents disturbances which increase with the time, the modulus of dicay or the coefficient of subsidence being k. For such disturbances the steady motion will be stable.

If the biquadratic has a pair of imaginary roots of the form $\alpha+\beta i$, the corresponding disturbance take the form $e^{\alpha t}$ $(a\cos\beta t+b\sin\beta t)$ and if the real part α be positive this presents an oscillation which increases with the time and steady motion will be unstable. If on the other hand the real part α is negative and equal to $-\gamma$, the solution takes the form $e^{-\gamma t}(a\cos\beta t+b\sin\beta t)$ and the disturbance becomes a damped oscillations of which the coefficient of subsidence is equal to γ. For such disturbance the system tends to revert to its state of steady motion and is stable.

The condition that the roots of the biquadratic equation shall all have their real part negative and thus indicate stability of steady motion is given by Routh in his Advanced Rigid Dynamics. Supposing A_0 positive as it is in the above equation this condition requires that A_0, B_0, C_0, D_0, E_0 and F_0 where

$$F_0=B_0C_0D_c-A_0D_0^2-E_0B_0^2 \text{ shall all be positive.}$$

The first group represents oscillations in the plane of x, y, which we call longitudinal or symmetrical oscillations ; the second group represents rotations p, q about the axes of x, and y and the motion w which we shall describe as lateral or transverse oscillation. This group consists of

$$C\frac{dw}{dt}+wz_w+pz_p+(AU+Z_q)q=-Z_0$$

$$P\frac{dp}{dt}+pL_p+qL_q+wL_w=-L_0-cW\cos\theta\sin\phi,$$

$$Q\frac{dq}{dt}+qM_q+pM_p+(\overline{C-A}+M_w)w=-M_0.$$

Subtituting from the equations of equilibrium and knowing $\theta=\theta_0+t$ and ϕ a small quantity

$$\left(C\frac{d}{dt}+Z_w\right)w+pZ_p+(AU+Z_q)q=0$$

$$\left(P\frac{d}{dt}+L_p\right)p+qL_q+wL_w=-cW\phi\cos\theta_0$$

$$\left(Q\frac{d}{dt}+M_q\right)q+pM_p+(C-A+M_w)w=0.$$

Let us assume p, q, w and ϕ each proportional to $e^{\lambda t}$ so that we have

$$\frac{dp}{dt}=\lambda p,\ \ \frac{dq}{dt}=\lambda q,\ \ \frac{dw}{dt}=\lambda w,\ \text{ and}$$

$$\lambda\phi\cos\theta_0=\frac{d\phi}{dt}\cos\theta_0=p\cos\theta_0-q\sin\theta_0$$

$$(c\lambda+Z_w)w+Z_p p+(AU+Z_q)pq=0$$

$$L_w w+(P\lambda+L_p)p+L_q q=-c\frac{W}{\lambda}(p\cos\theta_0-q\sin\theta_0)$$

$$(C-A+M_w)w+M_p p+(Q\lambda+M_q)q=0$$

or

$$(C\lambda+Z_w)w+Z_p p+(AU+Z_q)q=0$$

$$L_w w+(P\lambda+L_p+c\frac{W}{\lambda}\cos\theta_0)p+(L_q-c\frac{W}{\lambda}\sin\theta_0)q=0$$

$$(C-A+M_w)w+M_p p+(Q\lambda+M_q)q=0.$$

Hence eliminating w, p, q we get the following determinant

$$\begin{vmatrix} C\lambda+Z_w, & Z_p & AU+Z_q \\ L_w, & P\lambda+L_p+c\frac{W}{\lambda}\cos\theta_0 & L_q-c\frac{W}{\lambda}\sin\theta_0 \\ C-A+M_w, & M_p, & Q\lambda+M_q \end{vmatrix}=0$$

Expanding in powers of λ we get

$$A_1\lambda^4+B_1\lambda^3+C_1\lambda^2+D_1\lambda+E_1=0$$

where

$$A_1=CPQ$$

$$B_1=CPM_q+CQL_p+PQZ_w$$

$$\begin{aligned} C_1=&Z_wM_qP+L_pM_qC+Z_wL_pQ-M_pL_qC-L_wZ_pQ\\ &-P(AU+Z_q)(C-A+M_w)\\ =&P(Z_wM_q-Z_qM_w)+C(L_pM_q-M_pL_q)+Q(Z_wL_p-L_wZ_p)\\ &-APU(C-A+M_w)-P(AU+Z_q)(C-A)\end{aligned}$$

$$D_1 = cW\cos\theta_0(CM_q + Z_w Q) + CcM_p W\sin\theta_0 + Z_w(L_p M_q - L_q M_p)$$
$$+ L_w(M_p Z_q - M_q Z_p) + M_w(Z_p L_q - Z_q L_p)$$
$$+ (C-A)(Z_p L_q - Z_q L_p) + AU(L_w M_p - M_w L_p)$$
$$- AU(C-A)L_p$$

$$E_1 = cW\cos\theta_0(Z_w M_q - Z_q M_w) + cW\sin\theta_0(Z_w M_p - Z_p M_w)$$
$$- cW(C-A)(Z_p\sin\theta_0 + Z_q\cos\theta_0) - AU(C-A+M_w)cW\cos\theta_0$$

As before we have

$$F_1 = B_1 C_1 D_1 - A_1 D_1^2 - E_1 B_1^2.$$

The conditions of lateral stability are given by

$$A_1 B_1 C_1 D_1 E_1 F_1$$

being all positive ; A_1 being positive as is apparent from the equations.

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On the propagation of elastic waves in isotropic heterogeneous media*

By

Sudhansukumar Banerji.

1. *Introduction.*

In connection with the problem of seismic waves, it is of importance to consider the heterogeneity of the earth's structure. Even with the simplest assumption regarding the distribution of the density of the material composing the earth, the problem becomes so exceedingly complicated that it has been usual to regard the earth as isotropic in almost all theoretical discussion of the propagation of seismic waves. It is therefore thought that the few cases of wave propagation in medium having arbitrarily assumed laws of density considered in the following articles may not altogether be without interest.

2. *Two dimensional equations of motion.*

The equations of motion of an isotropic elastic solid in two dimensions are

$$\rho\frac{\partial^2 u}{\partial t^2}=(\lambda+\mu)\frac{\partial \Delta}{\partial x}+\mu\nabla^2 u$$

$$\rho\frac{\partial^2 v}{\partial t^2}=(\lambda+\mu)\frac{\partial \Delta}{\partial y}+\mu\nabla^2 v$$

where (u,v) are the component displacements, ρ is the density, λ,μ are the elastic constants, and

$$\Delta=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}.$$

These equations are satisfied by

$$u=\frac{\partial \phi}{\partial x}+\frac{\partial \psi}{\partial y},\quad v=\frac{\partial \phi}{\partial y}-\frac{\partial \psi}{\partial x},$$

* The present essay has been based on a method of deducing the motion of a heterogeneous from that of a homogeneous vibrating membrane given by Routh in the twelfth volume of the Proceedings of the London Mathematical Society, 1881.

provided

$$\frac{\partial^2\phi}{\partial t^2}=\frac{\lambda+2\mu}{\rho}\nabla^2\phi,\quad \frac{\partial^2\psi}{\partial t^2}=\frac{\mu}{\rho}\nabla^2\psi.$$

Consequently there are two types of waves in the medium travelling with the velocities $\sqrt{\frac{\lambda+2\mu}{\rho}}$ and $\sqrt{\frac{\mu}{\rho}}$ known respectively as irrotational and equivoluminal.

In the case of simple harmonic motion, the time factor being e^{ipt}, the latter equations take the forms

$$(\nabla^2+h^2)\phi=0,\quad (\nabla^2+k^2)\psi=0,$$

where $$h^2=\frac{p^2\rho}{\lambda+2\mu}=p^2a^2,\quad k^2=\frac{p^2\rho}{\mu}=p^2b^2,$$

the symbol denoting as in Lamb's paper* the *wave slowness, i.e.*, the reciprocal of the wave velocities, corresponding to the irrotational and equivoluminal types of disturbances respectively.

These formulae give the following expressions for the component stresses

$$\frac{X_x}{\mu}=\frac{\lambda}{\mu}\Delta+2\frac{\partial u}{\partial x}=-k^2\phi-2\frac{\partial^2\phi}{\partial y^2}+2\frac{\partial^2\psi}{\partial x\partial y},$$

$$\frac{X_y}{\mu}=\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}=2\frac{\partial^2\phi}{\partial x\partial y}-k^2\psi-2\frac{\partial^2\psi}{\partial x^2},$$

$$\frac{Y_y}{\mu}=\frac{\lambda}{\mu}\Delta+2\frac{\partial v}{\partial y}=-k^2\phi-2\frac{\partial^2\phi}{\partial x^2}-2\frac{\partial^2\psi}{\partial x\partial y}.$$

3. *Conjugate Transformations.*

We propose to prove in this article how by the use of the the theory of conjugate transformations it is possible to determine the wave propagation in medium having certain defined laws of distribution of density from the nature of wave propagation in the corresponding homogeneous medium which is supposed to be known.

If we have two variables ξ, η connected with x and y by the relation

$$\xi+\eta\sqrt{-1}=f(x+\sqrt{-1}),$$

* Lamb, "On the propagation of tremors over the surface of an elastic solid," *Phil. Trans.*, A, Vol. 203, 1904.

where f is any real functional symbol, we have

$$\frac{\partial \xi}{\partial x}=\frac{\partial \eta}{\partial y},\quad \frac{\partial \xi}{\partial y}=-\frac{\partial \eta}{\partial x},\quad \frac{\partial x}{\partial \xi}=\frac{\partial y}{\partial \eta},\quad \frac{\partial y}{\partial \xi}=-\frac{\partial x}{\partial \eta}.$$

We may also show by a simple transformation of variables that

$$\left(\frac{\partial \Delta}{\partial x}\right)^2+\left(\frac{\partial \Delta}{\partial y}\right)^2=\left\{\left(\frac{\partial \Delta}{\partial \xi}\right)^2 +\left(\frac{\partial \Delta}{\partial \eta}\right)^2\right\}\left\{\left(\frac{\partial \xi}{\partial x}\right)^2+\left(\frac{\partial \xi}{\partial y}\right)^2\right\},$$

$$\frac{\partial^2 \Delta}{\partial x^2}+\frac{\partial^2 \Delta}{\partial y^2}=\left\{\frac{\partial^2 \Delta}{\partial \xi^2}+\frac{\partial^2 \Delta}{\partial \eta^2}\right\}\left\{\left(\frac{\partial \xi}{\partial x}\right)^2+\left(\frac{\partial \xi}{\partial y}\right)^2\right\}.$$

Since x, y and ξ, η are interchangeable in this formula, it easily follows that

$$\left\{\left(\frac{\partial \xi}{\partial x}\right)^2+\left(\frac{\partial \xi}{\partial y}\right)^2\right\}\left\{\left(\frac{\partial x}{\partial \xi}\right)^2+\left(\frac{\partial x}{\partial \eta}\right)^2\right\}=1$$

It is also well-known that the angle made by any two curves which meet at a point P in the (x,y) plane is equal to the angle between the corresponding curves which meet at any point π in the (ξ,η) plane.

Suppose now that the nature of wave motion in two dimensional homogeneous medium with given boundary conditions are known, say for example, the dilatation in such a medium is given by $\Delta=\phi(\xi,\eta,t)$, where Δ represents the dilatation at a point whose coordinates are (ξ,η). Then this value of Δ satisfies the equation

$$\rho_0\frac{\partial^2 \Delta}{\partial t^2}=(\lambda+2\mu)\Delta_1^2\Delta.$$

Let (x,y) be the coordinates of a point in another medium. Let the density ρ of this heterogeneous medium be given by

$$\frac{\rho}{\rho_0}=\left(\frac{\partial \xi}{\partial x}\right)^2+\left(\frac{\partial \xi}{\partial y}\right)^2.$$

Then the equation for dilatational waves for the new medium is

$$\rho\frac{\partial^2 \Delta}{\partial t^2}=(\lambda+2\mu)\left[\frac{\partial^2 \Delta}{\partial x^2}+\frac{\partial^2 \Delta}{\partial y^2}\right].$$

But since (ξ,η) are known functions of (x,y), we obtain by substitution in the equation $\Delta=\phi(\xi,\eta,t)$, the new relation $\Delta=\psi(x,y,t)$, say, which is the solution of the equation determining the dilatation in the heterogeneous medium.

It should be noted that the two mediums are so related that the masses of corresponding squares in the homogeneous and heterogeneous media are equal to each other. Consequently the whole masses of the two media (supposed to be finite) are the same but differently distributed. It may also be noticed that the curves whose equations are given by

$$\left(\frac{\partial \xi}{\partial x}\right)^2+\left(\frac{\partial \xi}{\partial y}\right)^2=\left(\frac{\partial \eta}{\partial x}\right)^2+\left(\frac{\partial \eta}{\partial y}\right)^2=\text{constant},$$

represent the curves of constant density.

It is not every two-dimensional heterogeneous medium whose motion can be deduced from that of a homogeneous one. If we eliminate ξ between

$$\left(\frac{\partial \xi}{\partial x}\right)^2+\left(\frac{\partial \xi}{\partial y}\right)^2=\frac{\rho}{\rho_0},\quad \frac{\partial^2 \xi}{\partial x^2}+\frac{\partial^2 \xi}{\partial y^2}=0,$$

we easily obtain

$$\frac{\partial^2 \log\rho}{\partial x^2}+\frac{\partial^2 \log\rho}{\partial y^2}=0,$$

and therefore

$$\frac{\partial^2 \log\rho}{\partial \xi^2}+\frac{\partial^2 \log\rho}{\partial \eta^2}=0.$$

The density of the heterogeneous medium must therefore be such that its logarithm satisfies Laplace's equation.

4. *Two dimensional problems.*

As an illustration of the method let (x,y) be the Cartesian coordinates, (r,θ) the polar coordinates of a point P on the heterogeneous medium; (ξ,η) the Cartesian, (R,Θ) the polar coordinates of the corresponding point π in the homogeneous medium. If the relation between the two points be taken to be

$$\xi+i\eta=n\log\frac{x+iy}{a},$$

then
$$\xi=n\log\frac{r}{a},\quad \eta=n\theta.$$

Thus straight lines in the homogeneous medium parallel to the axis of ξ correspond to straight lines in the heterogeneous medium which pass through the origin. At the same time straight lines parallel to the axis of η correspond to circles whose centre is at the origin.

The density ρ is given by

$$\frac{\rho}{\rho_0}=\left(\frac{\partial\xi}{\partial r}\right)^2+\left(\frac{\partial\xi}{r\partial\theta}\right)^2=\left(\frac{n}{r}\right)^2.$$

If r vanish, ρ is infinite, it will therefore be necessary to exclude the origin from the area of the medium. If then, we know the motion of a homogeneous two-dimensional medium bounded by a rectangle, the transformation immediately gives the motion of a heterogeneous medium bounded by two circular arcs and any two radii vectores.

Another useful relation between the corresponding points P and π is

$$\xi+i\eta=a\left(\frac{x+iy}{a}\right)^n.$$

This gives

$$\xi=a\left(\frac{r}{a}\right)^n\cos n\theta,\ \eta=a\left(\frac{r}{a}\right)^n\sin n\theta,$$

and therefore in polar coordinotes

$$\mathrm{R}=a\left(\frac{r}{a}\right)^n,\ \Theta=n\theta.$$

By this transformation all straight lines are turned round the origin and altered in a known manner. The density ρ of the heterogeneous medium is given by

$$\frac{\rho}{\rho_0}=n^2\left(\frac{r}{a}\right)^{2(n-1)}.$$

Since θ=constant makes Θ=constant, we see that straight lines through the origin correspond to straight lines through the origin. All circles whose centres are at the origin correspond to circles whose centres are at the origin. When $n=-1$, we get the ordinary case of inversion, thus

$$\mathrm{R}=\frac{a^2}{r},\quad \Theta=\theta.$$

In this case any circle inverts into a circle and the density of the medium is given by

$$\frac{\rho}{\rho_0}=\frac{a}{r}.$$

As this is infinite when $r=0$, the centre of inversion must be external to the medium.

As in Lamb's paper,* suppose that the vibrations of the homogeneous solid are due to prescribed forces at or near the plane $\eta=0$. Hence a typical solution for the region $\eta>0$, can be assumed to be

$$\phi=Ae^{-\alpha\eta}e^{iH\xi}, \quad \psi=Be^{-\beta\eta}e^{iH\xi},$$

where H is real, and α, β are the positive real, or positive imaginary quantities determined by

$$\alpha^2=H^2-h^2, \quad \beta^2=H^2-k^2.$$

For the region $\eta<0$, the corresponding assumption would be

$$\phi=A'e^{\alpha\eta}e^{iH\xi}, \quad \psi=B'e^{\beta\eta}e^{iH\xi}.$$

The time factor is here temporarily omitted

The former expressions give for the displacements and the stresses at the plane $\eta=0$,

$$u_0=(iHA-\beta B)e^{iH\xi}, \quad v_0=(-\alpha A-iHB)e^{iHx},$$

and
$$[X_\nu]_0=\mu\{-2iH\alpha A+(2H^2-k^2)B\}e^{iH\xi}$$

$$[Y_\nu]_0=\mu\{(2H^2-k^2)A+2iH\beta B\}e^{iH\xi}$$

For the case of a "semi-infinite" elastic solid, bounded (say) by the plane $\eta=0$ and lying on the positive side of this plane, we may take as a typical distribution of normal force

$$[X_\nu]_0=0, \quad [Y_\nu]_0=Ye^{iH\xi}.$$

The constants A and B are determined by

$$-2iH\alpha A+(2H^2-k^2)B=0,$$

$$(2H^2-k^2)A+2iH\beta B=\frac{Y}{\mu},$$

which give

$$A=\frac{2H^2-k^2}{F(H)}\cdot\frac{Y}{\mu}, \quad B=\frac{2iH\alpha}{F(H)}\cdot\frac{Y}{\mu},$$

where,
$$F(H)=(2H^2-k^2)^2-4H^2\alpha\beta.$$

Now making use of the transformation

$$\eta+c+i\xi=n\log\frac{x+iy}{a},$$

* l.c., pp. 7-10.

that is

$$\eta+c=n\log\frac{r}{a}, \qquad \xi=n\theta,$$

we solve the corresponding problem for a heterogeneous medium having the law of density $\left(\frac{n}{r}\right)^2$ and the same typical normal distribution acting on the circle of radius

$$ae^{\frac{c}{n}}.$$

As has been shown by Lamb, the effect of an internal source of disturbance, resident say in the line $\xi=0$, $\eta=f$, the boundary $\eta=0$ being entirely free may be easily calculated. The simplest type of source is one which would produce symmetrical radial motion (in two dimensions) in an untimited solid, say,

$$\phi=D_0(hR), \quad \psi=0,$$

where $R=\sqrt{\{\xi^2+(\eta-f)^2\}}$, denotes distance from the source. If we superpose on this an equal source in the line $\xi=0$, $\eta=-f$, we obtain

$$\phi=D_0(hR)+D_0(hR'), \ \psi=0,$$

where $R'=\sqrt{\{\xi^2+(\eta+f)^2\}}$. It is evident without calculation that the condition of zero tangential stress at the plane $\eta=0$ is already satisfied; the normal stress however, does not vanish. In the neighbourhood of the plane $\eta=0$, the preceding value of ϕ is equivalent to

$$\phi=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{e^{a(\eta-f)}}{a}\,e^{iH\xi}\,dH+\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{e^{-a(\eta+f)}}{a}\,e^{iH\xi}\,dH$$

$$=\frac{2}{\pi}\int_{-\infty}^{\infty}\frac{\cosh a\eta}{a}\,e^{-af}e^{iH\xi}\,d\xi.$$

This makes

$$[X_y]_0=0, \quad [Y_y]_0=\frac{2\mu}{\pi}\int_{-\infty}^{\infty}\frac{2\xi^2-k^2}{a}\,e^{-af}\,e^{iH\xi}\,d\xi.$$

From this it is seen that the desired condition of zero stress on the boundary is satisfied, provided we superpose on the above expressions for ϕ, ψ, the solution given by

$$\phi=\int_{-\infty}^{\infty} Ae^{-\alpha\eta}\, e^{iH\xi}\, dH, \quad \psi=\int_{-\infty}^{\infty} Be^{-\beta\eta}\, e^{iH\xi}\, dH$$

where
$$\alpha^2=H^2-h^2, \quad \beta^2=H^2-k^2$$

$$A=\frac{2H^2-k^2}{F(H)}\cdot\frac{Y}{\mu}, \quad B=\frac{2iH\alpha}{F(H)}\cdot\frac{Y}{\mu}.$$

$$F(H)=(2H^2-k^2)^2-4H^2\alpha\beta,$$

$$Y=-\frac{2\mu}{\pi}\,\frac{2H^2-k^2}{\alpha}\, e^{-\alpha f}$$

From the above solution for a homogeneous semi-infinite medium bounded by $\eta=0$, we can easily infer that the disturbance in an infinite heterogeneous medium bounded by two fixed radii vectores forming a corner of angle $\frac{\pi}{n}$ and acted on by a source at a point (r_1, θ_1) and having the law of density $\frac{\rho}{\rho_0}=n^2\left(\frac{r}{c}\right)^{2(n-1)}$ is given by

$$\phi=\{D_0(hN)+D_0(hN')\}, \quad \psi=0,$$

where

$$C^{2n-2}N^2=r^{2n}+r_1^{2n}-2r^n r_1^n \cos n(\theta-\theta_1),$$

$$C^{2n-2}N'^2=r^{2n}+r_1^{2n}-2r^n r_1^n \cos n(\theta+\theta_1),$$

(r,θ) denoting the running coordinates of any point in the heterogeneous medium, with a superimposed disturbance given by

$$\phi=\int_{-\infty}^{\infty} Ae^{-\alpha y}\, e^{iHx}\, dH, \quad \psi=\int_{-\infty}^{\infty} Be^{-\beta y}\, e^{iHx}\, dH,$$

where A and B have the same values as before and x, y denote a point in the heterogeneous medium and are connected with ξ, η by the relation

$$\xi+i\eta=c\left(\frac{x+iy}{c}\right)^{n}.$$

If in the above we make n slightly differ from unity, that is to say, put $n=1+\epsilon$, where ϵ is a small quantity, we get the solution for the case of a semi-infinite solid bounded by two straight lines inclined at a small angle $\pi(1-\epsilon)$ the law of density slightly differing from homogeneity.

5. *Three dimensional problems.*

It is not possible to obtain a transformation similar to the one above indicated for the treatment of three dimensional problems. For if ξ,η,ζ denote three orthogonal curvilinear coordinates, Laplace's equation is transformed into

$$\nabla^2\phi=h_1h_2h_3\left[\frac{\partial}{\partial\xi}\left(\frac{h_1}{h_2h_3}\frac{\partial\phi}{\partial\xi}\right)+\frac{\partial}{\partial\eta}\left(\frac{h_2}{h_3h_1}\frac{\partial\phi}{\partial\eta}\right)\right.$$
$$\left.+\frac{\partial}{\partial\zeta}\left(\frac{h_3}{h_1h_2}\frac{\partial\phi}{\partial\zeta}\right)\right],$$

where as usual

$$\frac{1}{h_1^2}=\left(\frac{\partial x}{\partial\xi}\right)^2+\left(\frac{\partial y}{\partial\xi}\right)^2+\left(\frac{\partial z}{\partial\xi}\right)^2$$
$$=\left(\frac{\partial\xi}{\partial x}\right)^2+\left(\frac{\partial\xi}{\partial y}\right)^2+\left(\frac{\partial\xi}{\partial z}\right)^2, \text{ etc.}$$

Consequently, the above equation will reduce to the form

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}=\left(\frac{\partial^2\phi}{\partial\xi^2}+\frac{\partial^2\phi}{\partial\eta^2}\right.$$
$$\left.+\frac{\partial^2\phi}{\partial\zeta^2}\right)\left[\left(\frac{\partial\xi}{\partial x}\right)^2+\left(\frac{\partial\xi}{\partial y}\right)^2+\left(\frac{\partial\xi}{\partial z}\right)\left(\frac{\partial\xi}{\partial z}\right)^2\right],$$

provided $h_1=h_2=h_3$

and

$$\nabla^2\xi = h_1 h_2 h_3 \frac{\partial}{\partial\xi}\left(\frac{h_1}{h_2 h_3}\right) = 0,$$

$$\nabla^2\eta = h_1 h_2 h_3 \frac{\partial}{\partial\eta}\left(\frac{h_2}{h_3 h_1}\right) = 0,$$

$$\nabla^2\zeta = h_1 h_2 h_3 \frac{\partial}{\partial\xi}\left(\frac{h_3}{h_1 h_2}\right) = 0.$$

It is obvious that the above conditions cannot be sumultaneously satisfied for the conditions are equivalent to saying that h_1, h_2, h_3 must be all independent of ξ, η, ζ.

But when there is a symmetry as usually happens in earthquake problems, a method of solution can be obtained from a different consideration. In such cases an infinite number of two-dimensional vibration-types are to be arranged uniformly in all azimuths about the axis of symmetry and then the mean taken. For a consideration of this point for a homogeneous medium, reference is made to Lamb's paper, already cited, pp. 28-29. When a heterogeneous medium is considered, the integrations involved in the process may not easily be evaluated.

37

Some Problems in the Theory of Numbers

By

Pandit Oudh Upadhyaya.

(1) *On the failure of Legendre's rule in Theory of Numbers.*

The algebraical identity $4X = Y^2 + (-)^{\frac{p-1}{2}} z^2$, where $X = \frac{x^{p-1}}{x-1}$, p being any prime and Y and Z are polynomials of degree $\frac{p-1}{2}$ and $\frac{p-3}{2}$ respectively and the sign of ambiguity is $+$ or -1 according as p is of the form $4n+1$ or $4n+3$ was given by Gauss. In connection with this identity Legendre gave the following rule :—Y may be found by expanding $2(x-1)^{\frac{p-1}{2}}$ by binomial theorem and reducing each co-efficient to its absolutely least residue (mod. p) ; when the value of Y is known, the value of Z may be determined from Gauss's identity."

In a previous communication to the Lond. Math. Soc., the writer has shown that the least value of p for which this rule does not hold is 41. The rule is also found not to hold for 47.

(2) *Imperimitive groups of the sixth degree.*

The number of impirimitive groups of the sixth degree has been differently given by Cajori, Miller and Burnside. The question arises which of these impirimitive groups are Abelian. It is found that the cyclical group of order 6 is the only Abelian group in question. The remaining 14 groups are not Abelian.

(3) *On a problem in theory of numbers considered by E. Lucas.*

Lucas's problem consists in finding a number of any number of digits whose squares end with the same digits as the number itself.

* E. N. Barisien noted that the squares of 625, 9376, 8212890625 end with the same digits respectively as the original numbers. The same problem was also treated by R. Vercellin in the same journal.

* Suppl. al Periodico di Mat. 13, 1909, 20-21.

* G. Wertheim determined the numbers with seven or fewer digits whose squares end with the same digits as the number.

† Lloyd Tanner stated and Laisant proved that 87109376 and 12890625 are the only numbers of 8 digits whose squares end with the same 8 digits.

‡ G. R. Perkins and A. Martin stated that all powers of numbers ending with 12890625 end with the same digits.

|| E. Lucas noted that the only numbers having the same final ten digits as their squares are those ending with ten zeros, nine zeros followed by 1, 8212890625 and 1787109376.

The object of this paper is to consider the general case and to give a general rule with the help of which this problem may be solved to any number of digits.

It is evident that the numbers of one digit whose squares end with the same digit as the numbers themselves are 5, 6 and 1. There cannot be any other number of one digit whose square ends with the same digit as the number. Therefore we have to consider only three cases. We shall take first 5, then 6 and 1 in the end.

Case I. Let us take that case first in which 5 is in the units place. It is at once evident that 5 is a number, which, when squared will have 5 in the end and the digit 2 will go to the Tens-place, we can find the next number in question in the following way. Let us suppose that x is a digit in the Tens-place. Then the number of two digits which satisfies the given condition is $10x+5$, which when squared, becomes $100x^2+100x+25$. Thus it is evident that whatever be the value of x it cannot effect the number in the Tens-place. The only number which will remain in the tens-place is 2 which was over when we squared the number in the units place. By the reasoning given above, it is clear that when 5 is the required number in the units place, then whatever number might be put in the ten's place, the number in the ten's place when squared must be 2.

In order to have the same digit in the Ten's place in the original number as the digit of the number when squared, we must put 2 in the Ten's place ; but this two is the digit which was over when we squared the original number. This reasoning is applicable to the general case as well. Hence we obtain the following general rule :—When 5 is in

* Anfangsgrunde der Tahlenlehre 1902, 151-3.
† Nouv. Cornep. Math. 5, 1879, 217, 1880, 43.
‡ Math. Miscellany, Flushing, 26, 28.
|| Theorie des Nombres p. 88.

the units place, then square the number and the next digit thus obtained (after squaring) is the next digit required. The following examples will moke its meaning clear.

$5^2=25$

The next number is 2

$\therefore$ The required number of two digits is 25

$(25)^2=625$

The next digit is 6

$\therefore$ The required number of three digits is 625

$(625)^2=390625$

$\therefore$ The next digit is 0.

$(0625)^2=390625$

$\therefore$ The required number of 5 digits is 90625.

By applying the same rule, we can carry this operation to any number of times and each time, we shall get a digit which should be attached to the former number thus obtained.

I have carried this operation in order to find the required number of 15 digits, but the rule is general and can be carried to any number of digits.

The required number of 15 digis is 259918212890625.

Case II. Let 6 be the digit in the units place

$$6^2=36$$

$\therefore$ 3 will be added in the Tens' place.

Let the required number in the Tens' place be x.

$\therefore$ $10x+6$ is the required number of two digits.

$$(10x+6)^2=100x^2+120x+36$$

$$=100x^2+100x+20x+36$$

$\therefore$ The digit in the Tens' place is $2x+3$.

$$\therefore\ 2x+3\equiv x(\text{mod } 10)$$

$$\text{or}\ x+3\equiv 0(\text{mod } 10)$$

$$\therefore\ x=10-3.$$

In the last equation, we get $(10-3)$; this 3 is very important. This is a number which was over when we squared 6. Similarly when we square the number of two digits, we would get some number on the hundred's place and that number should take the place of three.

This reasoning is a general one, Hence we obtain the following general rule :—If a number of n digits satisfying the given condition be known, then square that number and subtract from 10, the number on the $(n+1)$th place, the number thus obtained will be the required number.

The following method will make its meaning clear .—

$$6^2=36$$

$\therefore$ $10-3=7$ is the next number

$\therefore$ The required number of two digits is 76

$$(76)^2=5776$$

$\therefore$ The number of three digits is 376.

The rule is general and we can find the required number of any number of digits by repeating this method. The number of 14 digits is 600817809376.

Case III. Let the number in the units place be 1

$$1^2=1$$

Nothing is over in this case and therefore nothing has been added to the number in the Tens' place in this case.

Let the digit in the Tens' place be x

$\therefore$ The required number of two digits is $10x+1$

$$10x+1=100x^2+20x+1,$$

$\therefore$ $2x$ is the digit which remains in the Tens' place

$\therefore$ $2x=x(\text{mod } 10)$

$\therefore$ $x=0(\text{mod } 10)$

$\therefore$ The least value of x, in this congruence is 0

$\therefore$ This rule does not give any next digit except 0

$\therefore$ When is there one in the end there cannot be any other solution except a number of one digit only. Numbers ending with any number of zeros followed by 1 satisfy the given condition.

If a is such a number of n digits such that $a^2=10^n \times b+a$, we can find a digit A to annex at the left hand side of a in order to obtain the number in question, but then, we have to solve a congruence at every step and calculation becomes troublesome, but this method is also a general method and can be applied to solve the problem in question. The answer in both the cases will be same. The same problem for cubes will be considered in the next paper

(4) *On a problem considered by S. Ramanujan.*

In the Proceedings of the Cambridge Philosophical Society, Vol. 19, (1916) Ramanujan has considered it the different positive integral values of a, b, c and d for which all positive integers be expressed in the form ;

$$ax^2+by^2+cz^2+dw^2$$

In this paper I have considered a particular problem, namely, that if particular values are given to a,b,c,d, (say) 1, 125, 50, 50, what are the numbers which can always be presented in the given form.

Let it be supposed that p is a prime number and q a factor of $(p-1)$, then, as has been proved in the "Theory of numbers" by Mr. B. Mathews, it can be proved that there is an equation of degree q with rational co-efficients, each of whose roots is the sum of $\frac{p-1}{q}$ of the primitive roots of unity ; no such pth root occurring in more than one of the sums. This is an Abelian equation and its group is cyclical.* It has been proved there that there is an unique solution of this problem If the multiplication table is formed, it is found that a set of two simultaneous relations occur and they are sufficient to ensure that the equations expressing the product form a consistent multiplication table. One of these relations is found to be $x^2+125y^2+50z^2+50w^2=144p$, where x, y, z and w are integers, subject to the condition

$$x\equiv 0 \text{ (mod. 3)}$$

$$y\equiv 0 \text{ (mod. 3)}$$

$$z\equiv 0 \text{ (mod. 3)}$$

$$w\equiv 0 \text{ (mod. 3)}$$

and p is a prime of the form $5n+1$; and at present we are not concerned with the other relation

Thus it is evident that every number of the form $144p$ where p is a prime number of the form $5n+1$, can be represented in the given form.

* *Proc. L. M. S.*, 1917.

The following table contains a list of some numbers of the form $5n+1$, and x, y, z and w are subject to the following conditions :

$$x \equiv 0 \pmod{3}$$

$$y \equiv 0 \pmod{3}$$

$$z \equiv 0 \pmod{3}$$

$$w \equiv 0 \pmod{3}$$

Number.	x	y	z	w
4464	11	3	6	3
8784	1	3	12	—3
14544	29	3	9	6
18844	11	3	3	18
21744	4	12	6	6
26064	11	3	6	21
27504	41	3	9	12
30384	1	15	3	6
34704	16	12	12	12
36144	4	12	6	18
39024	31	3	3	24

Calcutta Mathematical Society

Report for the year 1921

1. The gentlemen named below were elected officers and other members of the Council for the year 1921 :—

President :

The Hon'ble Justice Sir Asutosh Mookerjee, Swaraswati, Kt., C.S.I., M.A., D.L., D.Sc., Ph.D., F.R.S.E.

Vice-Presidents :

Mr. Mahendranath Ray, C.I.E., M.A., B.L.
Dr. C. E. Cullis, M.A., Ph.D.
Dr. Syamadas Mookerjee, M.A., Ph.D.

Treasurer :

Rai Bahadur Abinaschandra Bose, M.A.

Secretary :

Dr. Sudhănsukumar Banerji, D.Sc.

Council :

Dr. D. N. Mallik, D.Sc., F.R.S.E.
Dr. S. C. Bagchi, LL.D., Bar-at-Law.
Dr. C. V. Raman, M.A., D.Sc.
Satischandra Bose, Esq., M.A.
Manmathanath Ray, Esq., M.A., B.L.
Narendrakumar Majumdar, Esq., M.A.
Phanindralal Gangooli, Esq., M.A., B.L.
Dr. Surendramohan Gangooli, D.Sc.
Dr. Bibhutibhusan Datta, D.Sc.

2. During the year 1921, six meetings were held and 40 papers were communicated.

3. Four issues of the Bulletin (namely, Vol. XI, No. 4; Vol. XII, Nos. 1, 2, 3) containing thirty-six original papers, reviews, notices and miscellaneous notes have been published in 1921.

The four numbers of the Bulletin issued during the year form a volume exceeding 280 pages. The contents range over a variety of subjects and may be classified as follows :—

I. Pure Mathematics.

(1) *Theory of Functions.*

(*a*) Maurice Fréchet. Sur divers modes de convergence d'une suite de fonctions d'une variable. Contents : (*i*) Introduction. (*ii*) Convergence relativement uniforme. (*iii*) Influence de la notion "d'ecart" sur la definitions de la convergence. (*iv*) Generalisation de la convergence en measure. (*v*) Convergence presque uniforme. (*vi*) Cas des functions measurables.

(*b*) Abanibhusan Datta. On an extension of Sonine's integral in Bessel functions. The author shows simultaneously with Prof. Nilcholson (Quar. Journ. Vol. 48, 1921) that the value of the integral

$$\int_0^\infty J_m(a_1 x)J_m(a_2 x)\ \dots\ J_m(a_n x)x^{-mn+2m+1}dx$$

is zero or different from zero according as $(a_1, a_2, \dots a_n)$ do not or do form the sides of a polygon. When $(a_1, \dots a_n)$ do form the sides of a polygon, the author gives a value of the integral when the integrand involves four Bessel functions and also points out several mistakes in Nicholson's paper.

(*a*) Abanibhushan Datta. Expressions for the product of Bessel functions in a series of Bessel functions. The author considers expansions of the form $J_\mu(z)J_\nu(z)=\sum_n a_n J_{n+\mu+\nu}$, when μ and ν have any real value.

(2) *Higher Algebra including matrices and determinants.*

(*a*) Haripada Datta. On some symmetric determinants.

(*b*) R. Vaidyanath Swami. Binary Commutative Algebras.

(*c*) Haripada Datta. On the theory of Continued fraction.

(*d*) Sasindrachandra Dhar. On the inverse of an undegenerate non-plural quadrate slope. The object of this paper is to show that the equation given by the skeleton matrices AX=I of the type $\pi,\pi\}\{\pi,\pi\}=\{\pi,\pi\}$ in which A is a given undegenerate non-plural

quadrate slope, by which is meant a quadrate slope in which all the index numbers are defferent and I, a unit matrix, has a unique solution in which X is also a non-plural quadrate slope.

(*e*) Haripada Datta. On the evaluation of some recurrents and bigradients and on the expansion of some functions as power series.

(*f*) Haripada Datta. On a theorem in determinants. On algebraic remainders. On a theorem in simultaneous equations and on the solutions of a set of simultaneous equations.

(*g*) Satischandra Chakrabarty. On the transformation of a general determinant into a continuant and a recurrent and a new method of solving simultaneous equations.

(3) *Geometry and Trigonometry.*

(*a*) S. Mukhopàdhyay and G. Bhar. Generalisation of certain theorems in the hyperbolic geometry of the triangle. The authors discuss in this paper the analogues on the hyperbolic plane with actual, ideal or improper vertices of the following two well-known theorems in Euclidean Geometry :—(*i*) The three internal bisectors of the angles of a triangle or two external and one internal bisectors meet at a point. (*ii*) The three perpendiculars on the sides of a triangle from the opposite vertices meet at a point.

(*b*) G. H. Bryan. Graphic solution of spherical triangl
applications to astronomy. The author shows that the s
spherical triangle by constructive geometry involv
than the construction of a f

(*c*) C. V. Hanumanto
geneous co-ordinates.

(*d*) C. V. Hanumanto I

(*a*) Pandit

II. MATHEMATICAL PHYSICS.

(*a*) Sudhansukumar Banerji. Seismograms and their Significance. Contents: (*i*) Introduction. (*ii*) The three principal types of wave motion in an isotropic elastic solid. (*iii*) Division of the seismogram into three phases. (*iv*) Surface reflexions of earthquake waves. (*v*) Energy carried away by the three types of waves. (*vi*) Oceanic disturbance produced by submarine earthquakes. (*vii*) Interior of the Earth. (*viii*) Microseisms. (*ix*) Earthquake Periodicity.

(*b*) Sasindrachandra Dhar. On the convergence of the solutions of the second kind of Mathieu's equation in mathematical physics. In this paper the author gives a method of constructing the series of integrals of Mathieu's equation different from that given by Lindsay Ince (*Proc. Edin. Math. Soc.*, Vol. XXXIII, 1914-15). This series of integrals is found to be suitable for the consideration of its convergency.

(*c*) Sudhansukumar Banerji. On spherical Waves of Finite Amplitude produced by the sudden explosion of a detonating gas contained within a spherical envelope. It is shown that if P be the impulsive ressure at any point communicated by the explosion then P satisfies ıe equation $\frac{\partial P}{\partial t} = c^2 \nabla^2 P$ which is the same as the equation for ...nduction of heat and therefore the problem can be solved by ...vell-known methods in the Conduction of Heat.

...nan Das. On the scattering of light by molecules ... Strutt recently described an experiment which ... ctually the molecules of the air ... producing the blue colour of ... The object of this paper is to ... ıe intensity of light scattered

... a gravitating sphere.

... xists inside a

the magnetic field and that these terms are responsible for the asymmetry observed by Gmelin and Dufour, secondly, the author describes an atomic model which explains Runge's law of multiple resolution. Thirdly, the author shows that an extension of the dynamics of the $+H_2$ ion, leads to the result, that a part of the secondary spectrum of hydrogen should not show Zeeman effect. Lastly the author investigates the dynamical features of a hydrogen atom in a magnetic field taking relativity into account.

(*g*) Sudhansukumar Banerji. Percussion Figures in Isotropic Elastic Solids. The author gives an explanation of the rupture taking place in an isotropic elastic solid under the stresses set up by impact when these exceed the limit of perfect recovery observed by Raman (*Nature*, October 9, 1919) on the hypothesis that the maximum difference of the greatest and least principal stresses is a measure of the tendency to rupture.

(*h*) Sudhansukumar Banerji. On the solution of the equation $\nabla^4 \psi=0$ in bipolar coordinates.

(*i*) Brojendranath Chuckerbatti. On the distortion of the rings and brushes as observed through a twin crystal. Contents :—

(*j*) Introduction. (*ii*) Optical behaviour of a spath hemitrope. (*iii*) Explanation of the photographs. (*iv*) Physical Theory of Distortion.

(*k*) Panchanan Das. On the polarisation and intensity in the Complex Zeeman effect. An attempt to diseuss the question of the polarisation and intensity distribution of the components as well as the limit to the number of these components are made in this paper on Bohr's correspondence principle.

(*l*) Bholanath Pal. On the application Mathieu functions to Physical problems. Certain properties of Mathieu functions have been developed in this paper and these functions have been used to solve the problem of diffraction of waves by a screen of finite width and a finite rectilinear slit.

(*m*) Sudhansukumar Banerji. Notes on spherical waves of finite amplitude.

It is shown that spherical waves of finite amplitude are propagated with velocities

$$\left|\frac{1}{\rho}\,\frac{\frac{\partial p}{\partial r}}{\frac{\partial\, log\, \rho r^2}{\partial r}}\right|^{\frac{1}{2}}+u \text{ and } -\left|\frac{1}{\rho}\,\frac{\frac{\partial p}{\partial r}}{\frac{\partial\, log\, \rho r^2}{\partial r}}\right|^{\frac{1}{2}}+u.$$

4. The Society takes this opportunity to record its gratefulness to its distinguished President for his unabating zeal and sympathy for its development. The Society also tenders its best thanks to the other officers and the members of the Council for the valuable services rendered by them to the Society. The best thanks of the Society are also due to the Superintendent of the University Press for the care with which he has printed the Bulletin.

5. The Society also conveys its grateful thanks to the many learned Societies and Institutions and Publishers who have kindly presented a large number of books and journals to the Society's library.

6. Fifteen new members were elected during the year.

S. K. BANERJI

Secretary.

APPENDIX A

The following papers were read before the Calcutta Mathematical Society during the year 1921 :—

(1) Mr. Haripada Datta: "On the evaluation of some recurrents and bigradients and on the expansion of some functions as power series."

(2) Mr. Bholanath Pal: "On a treatment of the problem of diffraction of light by (*i*) a screen of finite width, (*ii*) by a rectilinear aperture on a screen by means of Mathieu functions."

(3) Mr. Panchanan Das: "Application of the Kinetic Theory of Gases to the problem of scattering of radiation."

(4) Mr. Oudh Upadhyay: "Imprimitive Groups of the Sixth Degree."

(5) Mr. Oudh Upadhyay: On a generalisation of a theorem by A. Cayley.

(6) Prof. S. K. Banerji: "On Spherical Waves of Finite Amplitude produced by the sudden explosion of a detonating gas contained within a spherical envelope."

(7) Mr. Sasindra Chandra Dhar: "On the Inverse of an undegenerate non-plural quadrate slope."

(8) Mr. Dhirendra Nath Mookerjee: "The Libration of colures and consequent changes in the commencement of the Hindu Nirayana Year."

(9) Mr. Panchanan Das: "On the abnormal Zeeman effect." (*Communicated by Prof. C. V. Raman.*)

(10) Mr. Hemanta Kumar Chakravarty: "On recurring decimals." (*Communicated by Prof. C. E. Cullis.*)

(11) Mr. Oudh Upadhyaya: "On a problem in the theory of numbers considered by E. Lucas."

(12) Mr. Nripendra Kumar Majumdar: "Disturbance produced by collision in an isotropic elastic solid."

(13) Prof. S. K. Banerji: "Oceanic disturbance produced by submarine earthquakes."

(14) Prof. G. H. Bryan, Sc.D., F.R.S.: "Graphic Solution of Spherical Triangles with applications to astronomy."

(15) Mr. Vaidyanath Swami, M.A.: "Binary Commutative Algebras." (*Communicated by the Secretary.*)

(16) Prof. S. K. Banerji, D.Sc.: "Percussion Figures in Isotropic Elastic Solids."

(17) Mr. Haripada Datta, M.A.: "(*i*) On a theorem in determinants; (*ii*) On algebraic remainders; (*iii*) On a theorem in simultaneous equations; and (*iv*) on the solutions of a set of simultaneous equations."

(18) Mr. Bholanath Pal, M.A.: "On continued fractions associated with Mathieu's differential equation."

(19) Mr. Abanibhusan Datta, M.A.: "Expressions for the product of two Bessel functions."

(20) Mr. Panchanan Das, M.Sc.: "On the polarisation and intensity in the complex Zeeman-effect."

(21) Mr. Panchanan Das, M.Sc.: "On the dynamics of intra-atomic electrons."

(22) Mr. B. N. Chakravarty, M.Sc.: "On the Distortion of the *Rings and Brushes* as observed through a twin crystal."

(23) Mr. Panchanan Das, M.Sc.: "On the distributed electron-orbits in an electromagnetic field."

(24) Mr. Satischandra Chakravarty, M.Sc.: "(i) On the transformation of a general determinant into a continuent and a recurrent. (ii) A new method to solve simultaneous equations."

(25) Pandit Oudh Upadhyaya: "On the failure of Legendre's rule in a problem in the theory of numbers. (ii) On a problem considered by S. Ramanujan."

(26) Prof. S. K. Banerji: "On the dissipation of energy from a vibrating membrane."

(27) Prof. G. H. Bryan, Sc.D., F.R.S.: "Note on the graphical solution of spherical triangles."

(28) Prof. C. V. Raman: "Quantum theory of light."

(29) Prof. S. K. Banerji: "On the depth of earthquake focus."

(30) Prof. S. K. Banerji: "On the solution of the equation $\nabla^4\psi=0$ in bipolar co-ordinates."

(31) Prof. C. V. Hanumanto Rao: "" Fundamental Relations in homogeneous co-ordinates."

(32) Prof. C. V. Hanumanto Rao: "On the ϕ conic of two conics."

(33) Mr. Bidhubhusan Ray: "The optical analogue of the whispering gallery effect."

(34) Mr. G. Bhar: "The Osculating Conic at Infinity."

(35) Mr. Oudh Upadhyaya: "On an algebraical identity."

(36) Mr. Oudh Upadhyaya: "On the values of polynomials in a transformation formula for $\frac{x^p-1}{x-1}$ where p is any prime number of the form $2n+1$.

(37) Mr. Nalinikanta Basu: "The stability of a dirigible baloon."

(38) Mr. Nripendranath Sen: "On some problems of tidal oscillations."

(39) Mr. B. N. Chakravarty: "The non-radiating electronic orbits and the normal Zeeman effect."

(40) Mr. P. Das: "Caustics formed by diffraction."

APPENDIX B

INCOME AND EXPENDITURE ACCOUNT, 1921

Income.

		Rs.	A.	P.
Opening Balance ...		610	8	0
Misc. Receipts ...		...		
Admission Fee ...		60	0	0
Subscriptions	Rs.			
For 1917 ...	6			
„ 1918 ...	7			
„ 1919 ...	54			
„ 1920 ...	262			
„ 1921 ...	726			
		1,055	0	0
Sale Proceeds of the Bulletin ...		38	0	0
Reprints ...		58	0	0
Advance from Secretary ...		8	2	6
TOTAL ...		1,829	10	6

Expenditure.

	Rs.	A.	P.
Books & Periodicals	243	7	6
Payment of Outstanding Bill of previous year ...	...		
Binding Charges ...	122	8	0
Printing of Bulletin			
For Papers ...	62	0	6
„ Blocks ...	4	0	0
Despatch of Bulletin	82	3	3
Printing & Stationery ...	28	1	6
Postage ...	17	10	0
Meeting Expenses...	62	12	6
Furniture ...	36	5	0
Misc. Charges ...	33	4	9
Repairs ...	1	10	6
Establishment ...	549	0	0
Closing Balance ...	586	11	0
TOTAL ...	1,829	10	6

ON SOME PROBLEMS OF TIDAL OSCILLATIONS*

BY

NRIPENDRANATH SEN.

[*Read 19th February, 1922.*]

The problem of tidal oscillations in canals and estuaries has engaged the attention of many eminent mathematicians, a detailed bibliography of whose works in this line has been given by Prof. Chrystal,[1] but up to now the problem has been solved for a small number of cases only. The object of the present paper is to present the first instalment of the results of my investigation on Tidal Oscillations in canals for several new cases. Almost all the solutions that appear in the paper have been obtained by me in their most general aspect in Bessel, Mathieu, Legendre and Hypergeometric Functions and I have shewn that many cases discussed by Professors Chrystal,[2] Lamb,[3] McCowan[4] and others[5] are deducible as particular cases of the general problems here studied.

1. Taking the origin on the undisturbed level and the axis of x parallel to the length of the canal, the equations of free tidal oscillations in canals of variable section are

$$\frac{\partial^2 \eta}{\partial t^2}=\frac{g}{b}\,\frac{\partial}{\partial x}\left(hb\frac{\partial \eta}{\partial x}\right) \qquad \dots \quad (1)$$

$$\eta=-\frac{1}{b}\,\frac{\partial}{\partial x}(hb\xi) \qquad \dots \quad (2)$$

$$\frac{\partial^2}{\partial t^2}(hb\xi)=ghb\,\frac{\partial}{\partial x}\left[\frac{1}{b}\,\frac{\partial}{\partial x}(hb\xi)\right] \qquad \dots \quad (3)$$

[1] Chrystal, "Hydrodynamical Theory of Seiches," Trans. R. S., Edinburgh, t. 41, pp. 647-49, 1905

[2] *Ibid*, pp. 599-649. Also "Some results in the Mathematical Theory of Seiches," Proc. R. S., Edinburgh, 25, 328, 1904.

[3] Lamb, 4th Edition, pp 267-69 (Hydrodynamics).

[4] McCowan, 'On the Theory of Long Waves,—" Phil. Mag. (5), 35, 250, 1892.

[5] Wedderburn, "On Long Waves." American Journal of Mathematics, 36, 211, 1914. S. Dasgupta, "Some Cases of Tidal Oscillations in canals of variable section." Bull. Cal. Math. Soc., 10, 105, 1918-19.

where $b=$ breadth of the canal at the surface at a distance x.

$h=$ depth over the width b

$\xi=$ time integral of the displacement past the plane x up to the time t

and $\eta=$ the tidal elevation above the equilibrium level.

In all cases discussed in the papers, it is supposed that the canal is symmetrical about the vertical plane through the x-axis.

Case I

2 Let $h=h_0\left(\frac{x}{a}\right)^r$ and $b=b_0\left(\frac{x}{a}\right)^n$, assuming that $\eta\propto\cos(\sigma t+\epsilon)$, we obtain from (1)

$$x^r\frac{d^2\eta}{dx^2}+(n+r)x^{r-1}\frac{d\eta}{dx}+\frac{\sigma^2a^r}{gh_0}\eta=0 \qquad \dots \quad (4)$$

Putting $z=\pm\frac{x^{1-\frac{r}{2}}}{1-\frac{r}{2}}$ where the lower sign is to be taken when $r>2$, we have from (3) $\qquad \dots \quad (5)$

$$\frac{d^2\eta}{dz^2}+\frac{2n+r}{2-r}\frac{d\eta}{zdz}+\frac{\sigma^2a^r}{gh_0}\eta=0 \qquad \dots \quad (6)$$

whence
$$\eta=z^{-\frac{(n+r-1)}{2-r}}\left\{AJ_{\frac{n+r-1}{2-r}}(kz)+B\,J_{-\frac{(n+r-1)}{2-r}}(kz)\right\}\times\cos(\sigma t+\epsilon) \qquad \dots \quad (7)$$

where $\qquad k^2=\frac{\sigma^2a^r}{gh_0}$.

(A) Since η is to be finite when $x=0$, evidently the first term in the right-hand side in (7) is to be retained when $\frac{n+r-1}{2-r}$ is positive and $\frac{r}{2}<1$. In that case

$$\eta=Az^{-\frac{n+r-1}{2-r}}J_{\frac{n+r-1}{2-r}}(kz)\cos(\sigma t+\epsilon) \qquad \dots \quad (8)$$

If the canal communicates with an open sea at its mouth

$$z=a_0 \text{ (say)}$$

where tidal oscillation of the type $\eta=C\cos(\sigma t+\epsilon)$ is maintained, then

$$\eta=C\left(\frac{z}{a_0}\right)^{-\frac{n+r-1}{2-r}}\frac{J_{\frac{n+r-1}{2-r}}(kz)}{J_{\frac{n+r-1}{2-r}}(ka_0)}\cos(\sigma t+\epsilon). \qquad \dots \quad (9)^1$$

If, however, the canal be closed at $z=a_0$, the admissible values of σ are given by $\frac{d\eta}{dz}=0$ when $z=a_0$

$$i.e. \qquad \frac{d}{dz}\left\{ z^{-\frac{n+r-1}{2-r}} J_{\frac{n+r-1}{2-r}}(kz)\right\}=0 \text{ when } z=a_0$$

$$i.e. \qquad J_{\frac{n+1}{2-r}}(ka_0)=0 \qquad \dots \quad (10)^2$$

The roots of the above equation can be calculated[3] and σ determined from the values of k by $\sigma^2=\frac{k^2gh_0}{a^r}$.

Also ridges and furrows are given in either case by

$$\frac{d\eta}{dz}=0, \text{ i.e } J_{\frac{n+1}{2-r}}(kz)=0$$

whence corresponding values of z and consequently the wave lengths can also be determined.

(B) The solution (7) fails when $r=2$ in which case $z=\infty$. To obtain solution for η in this case, we have from (4)

$$\frac{x^2d^2\eta}{dx^2}+(n+2)x\frac{d\eta}{dx}+k^2\eta=0 \text{ where } k^2=\frac{\sigma^2a^2}{gh_0}.$$

[1] This result has also been obtained by P. F. Ward, *vide* Phil. Mag. (Series 6), 25, 105, 1913.—"On the transverse vibrations of a rod of varying cross section" to which paper my attention was drawn long after the present paper had been completed and read before the Calcutta Mathematical Society.

[2] See Whittaker, Mod. Analysis, p. 354.

[3] See Gray and Mathews, Ch. V, pp. 46-49.

Putting $y=\log x$ in the above equation, we have

$$\frac{d^2\eta}{dy^2}+(n+1)\frac{d\eta}{dy}+k^2\eta=0$$

whence

$$\eta=x^{-\left(\frac{n+1}{2}\right)}\left(\mathrm{A}x^{\sqrt{\left(\frac{n+1}{2}\right)^2-k^2}}+\mathrm{B}x^{-\sqrt{\left(\frac{n+1}{2}\right)^2-k^2}}\right)\cos(\sigma t+\epsilon) \quad \ldots \quad (11)$$

when $\left(\frac{n+1}{2}\right)^2-k^2>0$

$$=\mathrm{C}x^{-\left(\frac{n+1}{2}\right)}\cos(m\log x+a)\cos(\sigma t+\epsilon) \quad \ldots \quad (12)$$

when $\left(\frac{n+1}{2}\right)^2-k^2<0=-m^2$ (say)

$$=x^{-\frac{n+1}{2}}(\mathrm{A}+\mathrm{B}\log x)\cos(\sigma t+\epsilon) \quad \ldots \quad (13)$$

$$\text{when} \quad \left(\frac{n+1}{2}\right)^2-k^2=0.$$

From (11), (12) and (13) η is finite at $x=0$ only when $\frac{n+1}{2}\leq 0$. The ridges and furrows will be given by $\frac{d\eta}{dx}=0$ in all these cases and points of zero elevation by $\eta=0$. When $n+1>0$, η becomes infinitely large at the origin and wave elevation becomes greater and greater from the sea inwards. But owing to the assumption that η is not sufficiently large, the above equations will not represent the exact state of affairs that will take place at the origin. It is interesting to note that in (11) and (13) unless A, B be of different signs, η gradually increases from the sea inwards and attains its biggest value at the origin when $n+1>0$. When A and B are of different signs, there is only one point at which η vanishes. But when (12) is the solution, there will be several waves with bigger and bigger values of η from the sea inwards.

(C) If again $r>2$
$$z=\frac{x^{-\left(\frac{r}{2}-1\right)}}{\frac{r}{2}-1} \quad \text{from (5)}$$
$$=\infty$$
when $x=0$.

Hence, in (8) using asymptotic[1] expansion of Bessel's function we have,

$$\eta=z^{+\frac{n+r-1}{r-2}-\frac{1}{2}}\left\{A\cos\left(z+\frac{n+r-1}{r-2}\frac{\pi}{2}-\frac{\pi}{4}\right)\right.$$
$$\left.+B\cos\left(z+\frac{n+r-1}{r-2}\frac{\pi}{2}-\frac{\pi}{4}\right)\right\}\cos(\sigma t+\epsilon) \text{ approximately,}$$

when z is very large. ... (14)

In order that η should be finite at the origin *i.e.* $x=\infty$, we have,

$$\frac{n+r-1}{r-2}-\frac{1}{2}<0$$

i.e. if
$$n<-\frac{r}{2} \qquad \text{... (15)}$$

On other hand if $n>-\frac{r}{2}$, evidently $\eta=\infty$ at the origin. If again $n=-\frac{r}{2}$, the value of η becomes ∞^0 and is therefore indeterminate. To obtain it, reverting to the equation (6), we get

$$\frac{d^2\eta}{dz^2}+z^2\eta=0\ \left(\because n=-\frac{r}{2}\right)$$

$$\therefore \eta=A\cdot\cos(kz+a)\cos(\sigma t+\epsilon).$$

Hence, we find that η is finite at the origin and there will be several simple harmonic ridges and furrows given by

$$\sin(kz+a)=0.$$

If the canal be closed the modes of oscillations may be obtained as in (A).

[1] Whittaker, Mod. Analysis, p. 362.

(D) When the canal is of length $2z_0$ and is symmetrical about the vertical plane $z=z_0$ also, then the motion in the first half of the canal will be determined by (7) or (8) if $r\neq 2$. It is evident that normal modes will fall into two classes. In the first of these η will have opposite values at corresponding points of the two halves of the canal and will therefore vanish at the centre $z=z_0$, the values of σ are then determined by

$$J_{\frac{n+r-1}{2-r}}(kz_0)=0 \qquad \ldots (16)$$

In the second class, the value of η is symmetrical with respect to the centre so that

$$\frac{d\eta}{dz}=0 \text{ at the middle } z=z_0$$

i.e.

$$J_{\frac{n+1}{2-r}}(kz_0)=0 \qquad \ldots (17)$$

The roots of (16) and (17) can be calculated as indicated in (10) and corresponding values of σ determined by the relation

$$\sigma^2=\frac{k^2gh_0}{a^r}.$$

When $r=2$, the corresponding modes may be investigated as above from (11), (12) and (13). In (13), the mode is determined by $k^2=\left(\frac{n+1}{2}\right)^2$. The condition $\frac{d\eta}{dx}=0$ or $\eta=0$ when $x=x_0$ (say) determines the relation between the arbitrary constants A and B.

3. A few special cases of interest which have already been worked out may be easily deduced from the results obtained above.

(*a*) When $r=0$, $n=1$, *i.e.*, the breadth varies as the distance from the end $x=0$ and the depth is uniform From (5) and (9)

$$\eta=C\frac{J_0(kx)}{J_0(ka)}\cos(\sigma t+\epsilon).$$

(*b*) If $n=0$, $r=1$, from (5) and (9), $\eta=C\dfrac{J_0(2kx^{\frac{1}{2}})}{J_0(2ka^{\frac{1}{2}})}\cos(\sigma t+\epsilon)$

(c) If $n=0$, $r=1$, but the bed slopes uniformly from either end to the middle, the canal is evidently symmetrical about $x=x_0$ where $2x_0$ is the length of the canal. From (5) and (9)

$$\eta=C\frac{J_0(2kx^{\frac{1}{2}})}{J_0(2ka^{\frac{1}{2}})}\cos(\sigma t+\epsilon)$$

where σ is given by (16) or (17), *i.e.* either

$$J_0(2kx_0^{\frac{1}{2}})=0 \text{ or, } J_1(2kx_0^{\frac{1}{2}})=0.$$

These results are given in Prof. Lamb's Hydrodynamics, Ed. IV, pp. 267-268.

(d) If $n=2$, $r=0$, *i.e.* if the horizontal section is a parabola and the depth is uniform, from (5) and (9),

$$\eta=C\left(\frac{x}{a}\right)^{-\frac{1}{2}}\frac{J_{\frac{1}{2}}(kx)}{J_{\frac{1}{2}}(ka)}\cos(\sigma t+\epsilon)=\frac{Ca}{x}\ \frac{\sin kx}{\sin ka}\cos(\sigma t+\epsilon)$$

If the canal be closed at $x=a$, the admissible values of σ are given by

$$J_{\frac{3}{2}}(ka)=0 \text{ from (10)}$$

i.e.
$$\frac{\sin ka}{ka}-\cos ka=0, \ i.e. \ \tan ka=ka.$$

(e) If $r=1$, $n=1$, *i.e.* if the bed and the surface slope uniformly, from (5) and (9),

$$\eta=C\left(\frac{x}{a}\right)^{-\frac{1}{2}}\frac{J_1(2kx^{\frac{1}{2}})}{J_1(2ka^{\frac{1}{2}})}\cos(\sigma t+\epsilon)$$

if the canal communicate with the sea at $x=a$.

(f) If $r=1$, $n=\frac{1}{2}$, from (5) and (9)

$$\eta=C\left(\frac{x}{a}\right)^{-\frac{1}{4}}\frac{J_{\frac{1}{2}}(2kx^{\frac{1}{2}})}{J_{\frac{1}{2}}(2ka^{\frac{1}{2}})}\cos(\sigma t+\epsilon)$$

if the sea be at $x=a$.

The last three results were obtained by Mr. Sasadhar Dasgupta in the Bulletin of the Calcutta Mathematical Society, Vol. X, No. 2, pp. 105-110, 1918-19.

Case II

4. If b=constant, and $h=h_0-c\cos m(x+a)$, *i.e.* if the bottom of the canal be undulatory while the breadth is uniform, then putting z for $h\xi$ in (3) of § 1, we have

$$\frac{\partial^2 z}{\partial t^2}=gh\frac{\partial^2 z}{\partial x^2}=m^2gh_0\left\{1-\frac{c}{h_0}\cos X\right\}\frac{\partial^2 z}{\partial X^2} \qquad \dots \quad (1)$$

where
$$X=m(x+a) \qquad \dots \quad (2)$$

Assuming $z=h\xi\propto\cos(\sigma t+\epsilon)$, we have from (1)

$$\frac{d^2z}{dX^2}+\frac{\sigma^2 z}{m^2gh_0(1-\frac{c}{h_0}\cos X)}=0$$

Now expanding $(1-\frac{c}{h_0}\cos X)^{-1}$ in terms of cosines of multiples of X, the above equation may be written as

$$\frac{d^2z}{dX^2}+\left\{\theta_0+2\sum_{n=1}^{\infty}\theta_n\cos nX\right\}z=0 \qquad \dots \quad (3)$$

where $\theta_0=\dfrac{\sigma^2}{m^2gh_0\left(1-\frac{c^2}{h_0^2}\right)^{\frac{1}{2}}}$ and $\theta_n=\dfrac{\sigma^2k^n}{m^2gh_0\left(1-\frac{c^2}{h_0^2}\right)^{\frac{1}{2}}}$.

where k is given by
$$\frac{2k}{1+k^2}=\frac{c}{h_0}. \qquad \dots \quad (4)$$

Since
$$\sum_0^{\infty}\theta_n=\frac{\sigma_2}{m^2gh_0\left(1-\frac{c^2}{h_0^2}\right)^{\frac{1}{2}}}\sum_0^{\infty}k^n$$

$$=\frac{\sigma^2\left\{\left(1-\frac{c}{h_0}\right)^{\frac{1}{2}}+\left(1+\frac{c}{h_0}\right)^{\frac{1}{2}}\right\}}{2m^2gh_0\left(1-\frac{c^2}{h_0^2}\right)^{\frac{1}{2}}\left(1-\frac{c}{h_0}\right)^{\frac{1}{2}}}=\text{a finite quantity.}$$

The equation (3) can be solved by Hill's method[1] of solving Mathieu equation. The above equation being slightly different from Hill's equation, it will be better to give the method of its solution. To solve it assume

$$z=e^{\mu X}\sum_{-\infty}^{\infty}b_n e^{niX} \qquad \dots \quad (5)$$

[1] Whittaker Mod. Analysis, p. 407, 1941.

Substituting [illegible] different powers of e^{iX} to [illegible]

$$(\mu+ni)^2 b_n + \sum_{m=-\infty}^{\infty} \theta_m \, b_{n-m} = 0 \quad (n = \ldots -2, \ldots).$$

If we eliminate the co-officients of b_n determinantally after dividing the typical equation by $\theta_0 - n^2$ to secure convergence, we obtain Hill's determinantal equation

$$\begin{vmatrix} \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{(i\mu+2)^2-\theta_0}{2^2-\theta_0}, & \frac{-\theta_1}{2^2-\theta_0}, & \frac{-\theta_2}{2^2-\theta_0}, & \frac{-\theta_3}{2^2-\theta_0}, & \frac{-\theta_4}{2^2-\theta_0} \\ \frac{-\theta_1}{1^2-\theta_0}, & \frac{(i\mu+1)^2-\theta_0}{1^2-\theta_0}, & \frac{-\theta_1}{1^2-\theta_0}, & \frac{-\theta_2}{1^2-\theta_0}, & \frac{-\theta_3}{1^2-\theta_0} \\ \frac{-\theta_2}{0^2-\theta_0}, & \frac{-\theta_1}{0^2-\theta_0}, & \frac{(i\mu)^2-\theta_0}{0^2-\theta_0}, & \frac{-\theta_1}{0^2-\theta_0}, & \frac{-\theta_2}{0^2-\theta_0} \\ \frac{-\theta_3}{1^2-\theta_0}, & \frac{-\theta_2}{1^2-\theta_0}, & \frac{-\theta_1}{1^2-\theta_0}, & \frac{(i\mu-1)^2-\theta_0}{1^2-\theta_0}, & \frac{-\theta_1}{1^2-\theta_0} \\ \frac{-\theta_4}{2^2-\theta_0}, & \frac{-\theta_3}{2^2-\theta_0}, & \frac{-\theta_1}{2^2-\theta_0}, & \frac{-\theta_1}{2^2-\theta_0}, & \frac{(i\mu-2)^2-\theta_0}{2^2-\theta_0} \end{vmatrix} = 0.$$

If we write $\Delta(i\mu)$ for the determinant, the equation determining μ is $\Delta(i\mu)=0$.

Now, proceeding exactly as in 19·42 of Prof. Whittaker's Modern Analysis and observing that

$$\Delta(i\mu) = \Delta_1(i\mu) \operatorname*{Lt}_{p=\infty} \prod_{n=-p}^{p} \left\{ \frac{\theta_0 - (i\mu-n)^2}{\theta_0 - n^2} \right\}$$

$$= -\Delta_1(i\mu) \frac{\sin\pi(i\mu - \sqrt{\theta_0}) \sin\pi(i\mu+\theta_0)}{\sin^2(\pi\sqrt{\theta_0})}$$

we have
$$\Delta(i\mu) = \Delta(0) - \frac{\sin^2 \pi i\mu}{\sin^2(\pi\sqrt{\theta_0})} = 0$$

or, $\sin^2 \pi i\mu = \Delta(0) \sin^2(\pi\sqrt{\theta_0})$ which determines μ.

Now taking equation (5), we have

$$z = h\xi = \frac{b_0}{c_0} e^{\mu X} \sum_{-\infty}^{\infty} c_n e^{niX} \cos(\sigma t + \epsilon) \qquad \ldots \quad (6)$$

$\ldots$ in $\Delta(i\mu)$ $\Big]$.

Fr... we have, $\eta = -\dfrac{\partial z}{\partial x}$ which at once gives η by (6). If a tidal oscillation $\eta = c\cos(\sigma t+\epsilon)$ be maintained at the sea $x=a$, we obtain

$$\eta = c\,\frac{\sum(\mu+ni)\,c_n e^{m(\mu+ni)(x+a)}}{\sum(\mu+ni)\,c_n e^{m(\mu+ni)(a+a)}}\cos(\sigma t+\epsilon). \qquad \ldots \quad (7)$$

Case III

5. Any linear differential equation of the form

$$\frac{d^2\eta}{dx^2}+P(x)\frac{d\eta}{dx}+Q(x)\eta=0$$

may be transformed into

$$\frac{d^2v}{dx^2}+F(x)v=0$$

by putting
$$\eta = v\,e^{-\frac{1}{2}\int^x P(x)dx}$$

where
$$F(x)=Q(x)-\frac{1}{2}\frac{dP(x)}{dx}-\frac{1}{4}\{P(x)\}^2.$$

Now if $-a\leq x\leq a$ and $F(x)$ satisfies Dirichlet's condition[1] in the range $-a\leq x\leq a$, then $F(x)$ can be expanded in Fourier's series

$$F(x)=a_0+2\sum_0^\infty\left\{a_n\cos\frac{n\pi x}{a}+b_n\sin\frac{n\pi x}{a}\right\}$$

where
$$a_0=\frac{1}{2a}\int_{-a}^{a}F(x)dx$$

$$a_n=\frac{1}{2a}\int_{-a}^{a}F(x)\cos\frac{n\pi x}{a}\,dx$$

$$b_n=\frac{1}{2a}\int_{-a}^{a}F(x)\sin\frac{n\pi x}{a}\,dx.$$

[1] Whittaker Mod. Analysis, p. 167, 9·22.

If $F(x)$ be an even function of x, then b_n, etc. will evidently vanish in the above expansion and the equation will be reduced to the form

$$\frac{d^2v}{dy^2}+\left\{\theta_0+2\sum_0^\infty\theta_n\cos ny\right\}v=0 \text{ where } y=\frac{\pi x}{a}$$

and
$$\theta_n=\frac{a}{2\pi^2}\int_{-a}^{a}F(x)\cos\frac{n\pi x}{a}\,dx.$$

The above equation is exactly similar to (3) of Art. 4 and can be solved by Hill's method as in Case II. The above method of treatment may be made use of in solving several interesting problems of tidal oscillations as follows :—

(1) If $h=h_0$ (say), $b=b_0\sqrt{1+\frac{x^2}{a^2}}$ *i.e.* when the canal is of uniform depth and the horizontal section is a hyperbola with the x-axis as its conjugate axis, we have (assuming $\eta\propto\cos(\sigma t+\epsilon)$) from (1) Art. I

$$\frac{1}{\left(1+\frac{x^2}{a^2}\right)^{\frac{1}{2}}}\frac{d}{dx}\sqrt{1+\frac{x^2}{a^2}}\frac{d\eta}{dx}+\frac{\sigma^2}{gh_0}\eta=0 \qquad \dots \quad (1)$$

Putting $x=a\sinh 2z$ in (1), we have

$\dfrac{d^2\eta}{dz^2}+\dfrac{\sigma^2a^2}{2gh_0}(1+\cosh 2z)\,\eta=0$ ... (2) which is the well-known Mathieu equation. This can again be put into the form

$$\frac{d^2\eta}{d\zeta^2}-\frac{\sigma^2a^2}{2gh_0}(1+\cos 2\zeta)\,\eta=0 \text{ writing } i\zeta \text{ for } z.$$

Now ζ being imaginary it is doubtful whether Hill's solution will be convergent or not. But if the length of the canal be $-a\leq z\leq a$, the equation (2) may be written as

$$\frac{d^2\eta}{dy^2}+\left(\theta_0+2\sum_1^\infty\theta_n\cos ny\right)\eta=0 \quad \dots \qquad \dots \quad (2)$$

$$\text{where } y = \frac{\pi z}{a} \text{ and } \theta_n = \frac{a}{2\pi^2}\int_{-a}^{a} \frac{\sigma^2 a^2}{2gh_0} (1 + \cosh 2z) \cos \frac{\pi n z dz}{a}$$

$$= \frac{a\sigma^2 a^2}{\pi^2 g h_0} \cdot \frac{(-)^n \sinh 2a}{2^2 + \frac{n^2\pi^2}{a^2}}$$

$$\theta_0 = \frac{a\sigma^2 a^2}{\pi^2 g h_0} \times \frac{\sinh 2a + 2a}{2^2}$$

The equation can be solved as before.

(2) If $h = h_0$, $b = b_0 \sqrt{\frac{x^2}{a^2} - 1}$, putting $x = a \cosh z$ where $0 \leq z \leq a$, we have

$$\frac{d^2\eta}{dy^2} + (\theta_0 + 2\sum_{1}^{\infty} \theta_n \cos ny)\, \eta = 0 \qquad \dots (3)$$

$$\text{where } y = z\frac{\pi z}{a},\ \theta_0 = \frac{\sigma^2 a^2 a}{\pi^2 g h_0}\ \frac{\sinh 2a - 2a}{2^2},\ \theta_r = \frac{\sigma^2 a^2 a}{g h_0 \pi^2}\ \frac{(-1)^r \sinh 2a}{2^2 + \frac{n^2\pi^2}{a^2}}$$

Expression for η may be obtained as before.

(3) If $b = b_0$, $h = h_0 \sqrt{1 + \frac{x^2}{a^2}}$ *i.e.* if the canal be of uniform breadth and its longitudinal section be a hyperbola with its conjugate on the undisturbed level of the canal, we have proceeding as before

$$\frac{d}{dx} \sqrt{1 + \frac{x^2}{a^2}}\ \frac{d\eta}{dx} + \frac{\sigma^2}{gh_0}\ \eta = 0.$$

Putting $\frac{x}{a} = \sinh z$ where $-a \leq z \leq a$, we have

$$\frac{d^2\eta}{dy^2} + (\theta_0 + 2\sum_{1}^{\infty} \theta_n \cos ny)\, \eta = 0$$

$$\text{where } y = \frac{\pi z}{a},\ \theta_0 = \frac{a\sigma^2 a^2}{\pi^2 g h_0} \sinh a$$

$$\text{and } \theta_n = \frac{(-)^n \sinh a}{1 + \frac{n^2\pi^2}{a^2}}\ \frac{\sigma^2 a^2 a}{\pi^2 g h_0}$$

(4) when $h = h_0 \sqrt{1 + \frac{x^2}{a^2}}$

$b = b_0 \sqrt{1 + \frac{x^2}{a^2}}$ etc. when both the horizontal and longitudinal sections of the canal are hyperbolas, we have,

$$\frac{1}{\sqrt{1 + \frac{x^2}{a^2}}} \frac{d}{dx} \sqrt{1 + \frac{x^2}{a^2}} \frac{d\eta}{dx} + \frac{\sigma^2}{gh_0} \eta = 0$$

Putting $\frac{x}{a} = \sinh z$, $\frac{d^2\eta}{dz^2} + \tanh z \frac{d\eta}{dz} + K^2 \cosh z \, \eta = 0$ where

$k^2 = \frac{\sigma^2 a^2}{gh_0}$.

Again putting $\zeta = \cosh^{\frac{1}{2}}(z) \, \eta$ in the above we have,

$$\frac{d^2\zeta}{dz^2} + F(z)\,\zeta = 0 \text{ where } F(z) = K_0^2 \cosh z - \frac{1}{4\cosh^2 z} - \frac{1}{4}.$$

Evidently $F(z)$ satisfies Dirichlet's conditions in the range $-a \leq z \leq a$ and therefore ζ can be obtained as before.

Case IV

6. If h=constant=h_0 (say), $b = b_0 \cosh m(x+a)$, then assuming

$$\eta \propto \cos(\sigma t + \epsilon)$$

we have from equation (2) Art. (1)

$$\frac{1}{\cosh m(x+a)} \frac{d}{dx} \cosh m(x+a) \frac{d\eta}{dx} + \frac{\sigma^2}{gh_0} \eta = 0.$$

Putting $\sinh m(x+a) = z$, the above equation may be written as

$$(1 + z^2) \frac{d^2\eta}{dz^2} + 2z \frac{d\eta}{dz} + k^2\eta = 0 \qquad \dots \quad (1)$$

where $k^2 = \frac{\sigma^2}{gk_0 m^2}$.

In order to make η finite when $z=0$, let us assume

$$\eta = \sum_{0}^{\infty} A_r\, z^r. \qquad \dots \quad (2)$$

Substituting this value of η in (1) and equating the co-efficients of z_r to zero

$$\frac{Ar+2}{Ar} = -\frac{k^2+r\,(r+1)}{(r+1)\,(r+2)}. \qquad \dots \quad (3)$$

If we begin with $r=0$, we have

$$\frac{A^2}{A_0} = -\frac{k^2}{1\cdot 2},\ \frac{A_4}{A_2} = -\frac{k^2+2.3}{3\cdot 4}\ etc.$$

$$\therefore \text{ From (2), } \eta = A_0\left[(1-\frac{k^2}{\lfloor 2}z^2+\frac{k^2\,(k^2+2\cdot 3)}{\lfloor 4}z^4\right.$$

$$\left.-\frac{k^2\,(k^2+2\cdot 3)\,(k^2+4\cdot 5)}{\lfloor 6}z^6+\text{etc.}\right]\cos(\sigma t+\epsilon) \qquad \dots \quad (4)$$

If however we begin with $r=1$, we have

$$\eta = A_1\left[z-\frac{k^2+1\cdot 2}{\lfloor 3}z^3+\frac{(k^2+1\cdot 2)\,(k^2+3\cdot 4)}{\lfloor 5}z^5\right.$$

$$\left.-\dots\text{etc.}\right]\cos(\sigma t+\epsilon) \qquad \dots \quad (5)$$

It may be easily shewn that both of these converge when $z<1$. When (4) is the solution, evidently η is symmetrical with respect to $z=0$ but (5) represents an oscillation such that η has opposite values at corresponding points on either side of $z=0$.

7. When h=constant=h_0 (say), $b=b_0 \sinh m\,(x+a)$, then assuming $\eta \propto \cos(\sigma t+\epsilon)$, we have,

$$(z^2-1)\frac{d^2\eta}{dz^2}+2z\frac{d\eta}{dz}+k^2\eta=0 \qquad \dots \quad (6)$$

where $z=\cosh m(x+a)$ and $k^2=\dfrac{\sigma^2}{gh_0\,m^2}$. It is easy to see that the above equation may be obtained by putting iz for z in (1) of Art. 6. But $\because\ z=\cosh m\,(x+a) \not< 1$ for which values the series in (4) and (5) of the last article are divergent, the assumption to be made for η must be a series in inverse powers of z.

Let, therefore, $\eta=\sum A_r z^{-r}\cos(\sigma t+\epsilon)$. ... (7)

Substituting this value of η in (6) and equating the co-efficient of A_r to zero, we have $A_0=A_2=A_4\ldots=0$

$$\frac{A_3}{A_1}=\frac{1\cdot 2}{k_2+2.3},\ \frac{A_5}{A_3}=\frac{3\cdot 4}{k^2+4.5}\ \text{etc.}\ \frac{A_{2r+1}}{A_{2r-1}}=\frac{(2r-1)\,2r}{k^2+(2r+1)\,2r}$$

$$\therefore \eta=A_1[z^{-1}+\frac{\lfloor 2}{k^2+2\cdot 3}z^{-3}+\frac{\lfloor 4}{(k^2+2\cdot 3)\,(k^2+4\cdot 5)}z^{-5}$$

$$+\ldots]\cos(\sigma t+\epsilon). \quad \ldots \quad (8)$$

If U_r, U_{r+1} be the rth and $(r+1)$th terms of the series in (8).

then $$\text{Lt}\left|\frac{U_r}{U_{r+1}}\right|=\text{Lt}\left|\frac{k^2+2r\,(2r+1)}{2r\,(2r-1)}z^2\right|>1 \text{ if } z=1.$$

But if $z=1$, $\text{Lt}\left|\frac{U_r}{U_{r+1}}\right|=1$ and $\text{Lt}\left(\frac{U_r}{U_{r+1}}-1\right)r=1$,

and $\text{Lt}\,\{r\left(\frac{U_r}{U_{r+1}}-1\right)-1\}\log r=\text{Lt}\,\frac{(k^2+2)\log r}{4r-2}=0<1.$

Hence the series and therefore η is finite everywhere except at $z=1$ *i.e.* at the inmost point of the canal. But as we have supposed that η is finite, the result obtained above may not give exact information as to what happens at the inmost point of the canal.

Case V

8. If $h=h_0\{1-\left(\frac{x}{a}\right)^2\}$, $b=b_0\left(\frac{x}{a}\right)$ *i.e.* if the longitudinal section of the canal be a parabola and the sides of the canal slope uniformly, we have assuming $\eta\propto\cos(\sigma t+\epsilon)$,

$$\frac{1}{x}\ \frac{d}{dx}\left(1-\frac{x^2}{a^2}\right)x\ \frac{d\eta}{dx}+\frac{\sigma^2}{gh_0}\eta=0$$

Putting $y=\frac{x^2}{a^2}$ in the above equation,

$$y\,(1-y)\frac{d^2\eta}{dy^2}+(1-2y)\frac{d\eta}{dy}+k^2\eta=0 \quad \ldots \quad (1)$$

$$\text{where } k^2 = \frac{\sigma^2 a^2}{4gh_0}$$

The equation (1) is the well-known hypergeometric equation, the solution of which may be written as

$$\eta = C\,F\,(a, b\,;\,c\,;\,y), \cos(\sigma t+\epsilon) \qquad \ldots \quad (2)$$

where $c=1$, $a+b=1$, $ab=-k^2$ *i.e* $a=\dfrac{1+\sqrt{1+4k^2}}{2}$, $b=\dfrac{1-\sqrt{1+4k^2}}{2}$

and $F\,(a, b, c, y)$

$$=1+\frac{a\cdot b}{1\cdot c}\,y+\frac{a(a+1)\,b(b+1)}{1\cdot 2\,c(c+1)}\,y^2+\frac{a(a+1)\,(a+2)\,b(b+1)\,(b+2)}{1\cdot 2\cdot 3\cdot c\cdot(c+1)\cdot(c+2)}\,y^3$$

$$+\text{ etc.} \qquad \ldots \quad (3)$$

The series is convergent[1] for $|\,y\,| < |$ *i.e.* for the canal $x<a$

If the canal communicates with a sea at $y=y_1$ (say), where a tidal oscillation $\eta=A\cos(\sigma t+\epsilon)$ is maintained, we have from (2)

$$\eta=A\,\frac{F\,(a, b\,;\,c\,;\,y)}{F\,(a, b\,;\,c\,;\,y_1)}\,\cos(\sigma t+\epsilon)$$

If the canal be closed at $y=y_1$, the possible modes of oscillation are given by $\dfrac{d\eta}{dx}=0$ when $y=y_1$.

Now from (2) and (3)

$$\eta=C\,F\,(a, b\,;\,c\,;\,y)\cos(\sigma t+\epsilon)$$

$$=C\,F\left(\frac{1+\sqrt{1+4k^2}}{2},\ \frac{1-\sqrt{1+4k^2}}{2}\,;\ 1\,;\ y\right)\cos(\sigma t+\epsilon).$$

[1] Mod. Analysis, p. 24.

$$\therefore \text{F}(a, b, c, y) = \text{F}\left(\frac{\sqrt{1+4k^2}-1}{2}+1, -\frac{\sqrt{1+4k^2}-1}{2}; 1; \tfrac{1}{2}-\frac{y'}{2} \right)$$

writing $\dfrac{1-y'}{2}$ for y

$$= \text{P}_{\frac{\sqrt{1+4k^2}-1}{2}}(y') \qquad \dots \quad (4)^{1}$$

Hence, possible values of σ are given by

$$\frac{d}{dy'} \text{P}_{\frac{\sqrt{1+4k^2}-1}{2}}(y') = 0 \text{ when } y' = 1-2y_1$$

The different values of $\dfrac{\sqrt{1+4k^2}-1}{2}$ satisfying the above equation can be calculated[2] and corresponding values of σ determined by the relation

$$\sigma^2 = \frac{4k^2 g h_0}{a^2}$$

9. The previous case may be generalised as follows :—If $h = h_0 \left(1-\dfrac{x^2}{a^2}\right)$ and $b = b_0 \left(\dfrac{x}{a}\right)^n$, we have assuming $\eta \propto \cos(\sigma t+\epsilon)$,

$$\frac{1}{x^n} \frac{d}{dx} x^n \left(1-\frac{x^2}{a^2}\right) \frac{d\eta}{dx} + \frac{\sigma^2}{g h_0} \eta = 0$$

Putting η' for ηx^{n-1} and y for $\dfrac{x^2}{a^2}$ in the above equation,

$$y(1-y)\frac{d^2\eta'}{dy^2} + \left\{\frac{3-n}{2} - \left(\frac{3-n}{2}+1\right)y\right\}\frac{dy'}{dy}$$

$$+\left(\frac{\sigma^2 a^2}{4 g h_0} + \frac{n-1}{2}\right)\eta' = 0 \quad \dots \quad (5)$$

[1] Modern analysis P. 276, 14·11.

[2] Bholanath Pal, "On the numerical calculation of the roots of $\dfrac{d}{d\mu} \text{P}_n^m(\mu) = 0$ regarded as equation in n"—Bul. Cal Math. Soc. IX, 85, 1917-18.

Defining a, b, c by the relations, $c=\frac{3-n}{2}$, $a+b=\frac{3-n}{2}$

$$\text{and } ab=-\left(\frac{\sigma^2 a^2}{4gh_0}+\frac{n-1}{2}\right) \qquad \dots \quad (6)$$

The equation (5) may be written as

$$y\,(1-y)\frac{d^2\eta'}{dy^2}+(c-\overline{a+b+1}y)\;\frac{d\eta'}{dy}-ab\,\eta'=0 \qquad \dots \quad (7)$$

The solution of the Hypergeometric equation (7) is

$$\eta'=\eta x^{n-1}=\text{A F}\left(a, b\,;\, c\,;\frac{x^2}{a^2}\right)\cos(\sigma t+\epsilon) \text{ if } n-1 \not> 0$$

$$i.e.\ \eta=\text{A}x^{1-n}\ \text{F}\left(a, b\,;\, c\,;\frac{x^2}{a^2}\right)\cos(\sigma t+\epsilon) \qquad \dots \quad (8)$$

$$\text{if } n-1 \leq 0$$

$$\text{Also[1]}\quad \eta'=\eta x^{n-1}=\text{A}\,(x^2)^{1-c}\ \text{F}\left(a-c+1, b-c+1\,;\,2-c\,;\frac{x^2}{a^2}\right)$$

$$\cos(\sigma t+\epsilon)$$

$$i.e.\quad \eta=\text{A F}\left(a-c+1, b-c+1\,;\,\frac{n+1}{2}\,;\frac{x^2}{a^2}\right)\cos(\sigma t+\epsilon) \qquad \dots \quad (9)$$

$$\left[\because c=\frac{3-n}{2}\right].$$

The (9) will furnish a solution unless n be a negative odd integer, so that both (8) and (9) may give η of course with the exceptions mentioned above. η is evidently finite everywhere except when $x=a$. If the canal communicates with a sea at $x=a_1<a$ where a tidal oscillation is maintained in the form $\eta=c\cos(\sigma t+\epsilon)$, the value of A in (8) and (9) can be determined as before.

[1] Whittaker Mod. Analysis, p. 280.

10. (*a*) It is easy to deduce the results of Art. 8 from (8) and (9). For putting $n=1$, we have from (6), $c=1$, $a+b=1$, $ab=-\dfrac{\sigma^2 a^2}{4gh_0}$

$$a-c+1=a,\ b-c+1=b,\ \frac{n+1}{2}=1$$

From either (8) or (9), we get

$$\eta=\mathrm{A}\ \mathrm{F}\left(a, b; c; \frac{x^2}{a^2}\right)\cos(\sigma t+\epsilon) \text{ where } a, b, c \text{ are given above.}$$

(*b*) If $n=0$, $h=h_0\left(1-\dfrac{x^2}{a^2}\right)$, $b=b_0$, we have from (6)

$$c=\tfrac{1}{2}.\ a+b=\tfrac{1}{2},\ ab=-\left(\frac{\sigma^2 a^2}{4gh_0}+\frac{0-1}{2}\right)=-\frac{(r-1)\ (r+2)}{4}$$

putting $\sigma^2 a^2=r\ (r+1)\ gh_0$

$$\therefore\ a=1+\frac{r}{2},\ b=\tfrac{1}{2}-\frac{r}{2},\ c=\tfrac{3}{2}$$

Also $$a-c+1=\frac{r}{2}+\tfrac{1}{2},\ b-c+1=-\frac{r}{2},\ 2-c=\tfrac{1}{2}$$

Hence ; (8) gives

$$\eta=\mathrm{A}x\mathrm{F}\left(1+\frac{r}{2}, \tfrac{1}{2}-\frac{r}{2}; \tfrac{3}{2}, \frac{x^2}{a^2}\right)\cos(\sigma t+\epsilon) \qquad \cdots \quad (10)$$

and (9) gives $\eta=\mathrm{A}\ \mathrm{F}\left(\dfrac{r}{2}+\tfrac{1}{2}, -\dfrac{r}{2}; \tfrac{1}{2}; \dfrac{x^2}{a^2}\right)\cos(\sigma t+\epsilon)$. ... (11)

Both of these are solutions[1] of Legendre's equation and are convergent when $x<a$.

When r is integral, both of these series terminate and the solutions may be written as $\eta=\mathrm{C}\ \mathrm{P}_r\left(\dfrac{x}{a}\right)\cos(\sigma t+\epsilon)$.

[1] See Prof. Lamb's Hydrodynamics, Ed. IV, p. 106, Art. 84.

It is easy to deduce values of ξ from the corresponding values of η in (10) and (11) and the values of ξ thus obtained may be identified with Prof. Chrystal's Seiche Cosine and Sine functions respectively.

The results deduced in 10(b) from our general solution given in Art. 9 are identical with those already found out by Profs. Chrystal[1] and Lamb.[2]

[1] Prof. Chrystal, "Hydrodynamical Theory of Seiches" l.c.

[2] Prof Lamb, Hydrodynamics, Ed. IV, p. 269.